

A Simplified Characterisation of Provably Computable Functions of the System ID_1 of Inductive Definitions (Technical Report)

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Abstract. We present a simplified and streamlined characterisation of provably total computable functions of the theory ID_1 of non-iterated inductive definitions. The idea of the simplification is to employ the method of operator-controlled derivations that was originally introduced by Wilfried Buchholz and afterwards applied by the second author to a characterisation of provably total computable functions of Peano arithmetic PA.

Keywords: Provably Computable Functions; System of Inductive Definitions; Ordinal Notation Systems; Operator Controlled Derivations.

1 Introduction

As stated by Gödel's second incompleteness theorem, any reasonable consistent formal system has an unprovable Π_2^0 -sentence that is true in the standard model of arithmetic. This means that the total (computable) functions whose totality is provable in a consistent system, which are known as *provably computable functions* or *provably total functions*, form a proper subclass of total computable functions. It is natural to ask how we can describe the provably total functions of a given system. Not surprisingly provably (total) computable functions are closely related to provable well-ordering, i.e., *ordinal analysis*. Up to date ordinal analysis for quite strong systems has been accomplished by M. Rathjen [13,14] or T. Arai [1,2]. On the other hand several successful applications of techniques from ordinal analysis to characterisations of provably computable functions have been provided by B. Blankertz and A. Weiermann [4], W. Buchholz [7], Buchholz, E. A. Cichon and Weiermann [8], M. Michelbrink [10], or G. Takeuti [16]. Surveys on characterisations of provably computable functions of fragments of Peano arithmetic PA contain the monograph [9] by M. Fairtlough and S. S. Wainer.

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Modern ordinal analysis is based on the method of *local predicativity*, that was first introduced by W. Pohlers, c.f. [11,12]. Successful applications of local predicativity to provably computable functions contain works by Blankertz and Weiermann [18] and by Weiermann [5]. However, to the authors' knowledge, the most successful way in ordinal analysis is based on the method of *operator-controlled derivations*, an essential simplification of local predicativity, that was introduced by Buchholz [6]. In [19] the second author successfully applied the method of operator-controlled derivations to a streamlined characterisation of provably computable functions of PA. (See also [12, Section 2.1.5].) Technically this work aims to lift up the characterisation in [19] to an impredicative system \mathbf{ID}_1 of non-iterated inductive definitions. We introduce an ordinal notation system $\mathcal{O}(\Omega)$ and define a computable function f^α for a starting number-theoretic function $f : \mathbb{N} \rightarrow \mathbb{N}$ by transfinite recursion on $\alpha \in \mathcal{O}(\Omega)$. The ordinal notation system $\mathcal{O}(\Omega)$ comes from a draft [20] of the second author and the transfinite definition of f^α comes from [19]. We show that a function is provably computable in \mathbf{ID}_1 if and only if it is a Kalmar elementary function in $\{s^\alpha \mid \alpha \in \mathcal{O}(\Omega) \text{ and } \alpha < \Omega\}$, where s denotes the successor function $m \mapsto m + 1$ and Ω denotes the least non-recursive ordinal. (Corollary 42)

2 Preliminaries

In order to make our contribution precise, in this preliminary section we collect the central notions. We write \mathcal{L}_{PA} to denote the standard language of first order theories of arithmetic. In particular we suppose that the constant 0 and the successor function symbol S are included in \mathcal{L}_{PA} . For each natural m we use the notation \underline{m} to denote the corresponding numeral built from 0 and S . Let a set variable X denote a subset of \mathbb{N} . We write $X(t)$ instead of $t \in X$ and $\mathcal{L}_{\text{PA}}(X)$ for $\mathcal{L}_{\text{PA}} \cup \{X\}$. Let $\text{FV}_1(A)$ denote the set of free number variables appearing in a formula A and $\text{FV}_2(A)$ the set of free set variables in A . And then let $\text{FV}(A) := \text{FV}_1(A) \cup \text{FV}_2(A)$. For a fresh set variable X we call an $\mathcal{L}_{\text{PA}}(X)$ -formula $\mathcal{A}(x)$ a *positive operator form* if $\text{FV}_1(\mathcal{A}(x)) \subseteq \{x\}$, $\text{FV}_2(\mathcal{A}(x)) = \{X\}$, and X occurs only positively in \mathcal{A} .

Let $\text{FV}_1(\mathcal{A}(x)) = \{x\}$. For a formula $F(x)$ such that $x \in \text{FV}_1(F(x))$ we write $\mathcal{A}(F, t)$ to denote the result of replacing in $\mathcal{A}(t)$ every subformula $X(s)$ by $F(s)$. The language $\mathcal{L}_{\text{ID}_1}$ of the *theory \mathbf{ID}_1 of non-iterated inductive definitions* is defined by $\mathcal{L}_{\text{ID}_1} := \mathcal{L}_{\text{PA}} \cup \{P_{\mathcal{A}} \mid \mathcal{A} \text{ is a positive operator form}\}$ where for each positive operator form \mathcal{A} , $P_{\mathcal{A}}$ denotes a new unary predicate symbol. We write $\mathcal{T}(\mathcal{L}_{\text{ID}_1}, \mathcal{V})$ to denote the set of $\mathcal{L}_{\text{ID}_1}$ -terms and $\mathcal{T}(\mathcal{L}_{\text{ID}_1})$ to denote the set of closed $\mathcal{L}_{\text{ID}_1}$ -terms. The axioms of \mathbf{ID}_1 consist of the axioms of Peano arithmetic PA in the language $\mathcal{L}_{\text{ID}_1}$ and the following new axiom schemata (ID₁) and (ID₂):

- (ID₁) $\forall x(\mathcal{A}(P_{\mathcal{A}}, x) \rightarrow P_{\mathcal{A}}(x)).$
- (ID₂) (The universal closure of) $\forall x(\mathcal{A}(F, x) \rightarrow F(x)) \rightarrow \forall x(P_{\mathcal{A}}(x) \rightarrow F(x)),$
where F is an $\mathcal{L}_{\text{ID}_1}$ -formula.

For each $n \in \mathbb{N}$ we write IS_n to denote the fragment of Peano arithmetic PA with induction restricted to Σ_n^0 -formulas. Let k be a natural number and $f : \mathbb{N}^k \rightarrow \mathbb{N}$ a number-theoretic function and T be a theory of arithmetic containing IS_1 . Then we say f is *provably computable in T* or *provably total in T* if there exists a Σ_1^0 -formula $A_f(x_1, \dots, x_k, y)$ such that the following hold:

1. $\text{FV}(A_f) = \text{FV}_1(A_f) = \{x_1, \dots, x_k, y\}$.
2. For all $\mathbf{m}, n \in \mathbb{N}$, $f(\mathbf{m}) = n$ holds if and only if $A_f(\mathbf{m}, n)$ is true in the standard model \mathbb{N} of PA.
3. $\forall \mathbf{x} \exists! y A_f(\mathbf{x}, y)$ is a theorem in T .

It is well known that the provably computable functions of the theory IS_1 coincide with the primitive recursive functions. It is also known that the provably computable functions of the theory IS_2 coincide with the P  ter's multiply recursive functions.

3 A non-recursive ordinal notation system $\mathcal{OT}(\mathcal{F})$

In this section we introduce a *non-recursive* ordinal notation system $\mathcal{OT}(\mathcal{F}) = \langle \mathcal{OT}(\mathcal{F}), < \rangle$. This new ordinal notation system is employed in the next section. For an element $\alpha \in \mathcal{OT}(\mathcal{F})$ let $\mathcal{OT}(\mathcal{F}) \upharpoonright \alpha$ denote the set $\{\beta \in \mathcal{OT}(\mathcal{F}) \mid \beta < \alpha\}$.

Definition 1. We define three sets $\text{SC} \subseteq \mathbb{H} \subseteq \mathcal{OT}(\mathcal{F})$ of ordinal terms and a set \mathcal{F} of unary function symbols simultaneously. Let $0, \varphi, \Omega, S, E$ and $+$ be distinct symbols.

1. $0 \in \mathcal{OT}(\mathcal{F})$ and $\Omega \in \text{SC}$.
2. $\{S, E\} \subseteq \mathcal{F}$.
3. If $\alpha \in \mathcal{OT}(\mathcal{F}) \upharpoonright \Omega$, then $S(\alpha) \in \mathcal{OT}(\mathcal{F})$ and $E(\alpha) \in \mathbb{H}$.
4. If $\{\alpha_1, \dots, \alpha_l\} \subseteq \mathbb{H}$ and $\alpha_1 \geq \dots \geq \alpha_l$, then $\alpha_1 + \dots + \alpha_l \in \mathcal{OT}(\mathcal{F})$.
5. If $\{\alpha, \beta\} \subseteq \mathcal{OT}(\mathcal{F}) \upharpoonright \Omega$, then $\varphi\alpha\beta \in \mathbb{H}$.
6. If $\alpha \in \mathcal{OT}(\mathcal{F})$ and $\xi \in \mathcal{OT}(\mathcal{F}) \upharpoonright \Omega$, then $\Omega^\alpha \cdot \xi \in \mathbb{H}$.
7. If $F \in \mathcal{F}$, $\alpha \in \mathcal{OT}(\mathcal{F})$ and $\xi \in \mathcal{OT}(\mathcal{F}) \upharpoonright \Omega$, then $F^\alpha(\xi) \in \text{SC}$.
8. If $F \in \mathcal{F}$ and $\alpha \in \mathcal{OT}(\mathcal{F})$, then $F^\alpha \in \mathcal{F}$.

By definition $F(\xi) \in \mathcal{OT}(\mathcal{F})$ holds if $F^\alpha(\xi) \in \mathcal{OT}(\mathcal{F})$ for some $\alpha \in \mathcal{OT}(\mathcal{F})$. We write ω^α to denote $\varphi 0 \alpha$ and m to denote $\omega^0 \cdot m = \underbrace{\omega^0 + \dots + \omega^0}_{m \text{ many}}$.

Let Ord denote the class of ordinals and Lim the class of limit ones. We define a semantic $[\cdot]$ for $\mathcal{OT}(\mathcal{F})$, i.e., $[\cdot] : \mathcal{OT}(\mathcal{F}) \rightarrow \text{Ord}$. The well ordering $<$ on $\mathcal{OT}(\mathcal{F})$ is defined by $\alpha < \beta \Leftrightarrow [\alpha] < [\beta]$. Let Ω_1 denote the least non-recursive ordinal ω_1^{CK} . For an ordinal α we write $\alpha =_{NF} \Omega_1^{\alpha_1} \cdot \beta_1 + \dots + \Omega_1^{\alpha_l} \cdot \beta_l$ if $\alpha > \alpha_1 > \dots > \alpha_l$, $\{\beta_1, \dots, \beta_l\} \subseteq \Omega_1$, and $\alpha = \Omega_1^{\alpha_1} \cdot \beta_1 + \dots + \Omega_1^{\alpha_l} \cdot \beta_l$. Let ε_α denote the α th epsilon number. One can observe that for each ordinal $\alpha < \varepsilon_{\Omega_1+1}$ there uniquely exists a set $\{\alpha_1, \dots, \alpha_l, \beta_1, \dots, \beta_l\}$ of ordinals such that $\alpha =_{NF} \Omega_1^{\alpha_1} \cdot \beta_1 + \dots + \Omega_1^{\alpha_l} \cdot \beta_l$. For a set $K \subseteq \text{Ord}$ and for an ordinal α we will write $K < \alpha$ to abbreviate $(\forall \xi \in K) \xi < \alpha$, and dually $\alpha \leq K$ to abbreviate $(\exists \xi \in K) \alpha \leq \xi$.

Definition 2 (Collapsing operators).

1. Let α be an ordinal such that $\alpha =_{NF} \Omega_1^{\alpha_1} \cdot \beta_1 + \dots + \Omega_1^{\alpha_l} \cdot \beta_l < \varepsilon_{\Omega_1+1}$. The set $K_\Omega \alpha$ of coefficients of α is defined by

$$K_\Omega \alpha = \{\beta_1, \dots, \beta_l\} \cup K_\Omega \alpha_1 \cup \dots \cup K_\Omega \alpha_l.$$

2. Let $F : \text{Ord} \rightarrow \text{Ord}$ be an ordinal function. Then a function $F^\alpha : \text{Ord} \rightarrow \text{Ord}$ is defined by transfinite recursion on $\alpha \in \text{Ord}$ by

$$\begin{cases} F^0(\xi) = F(\xi), \\ F^\alpha(\xi) = \min\{\gamma \in \text{Ord} \mid \omega^\gamma = \gamma, K_\Omega \alpha \cup \{\xi\} < \gamma \text{ and} \\ (\forall \eta < \gamma)(\forall \beta < \alpha)(K_\Omega \beta < \gamma \Rightarrow F^\beta(\eta) < \gamma)\}. \end{cases}$$

Corollary 3. Let $F : \text{Ord} \rightarrow \text{Ord}$ be an ordinal function. Then $F^\beta(\eta) < F^\alpha(\xi)$ holds if one of the following holds.

1. $\beta < \alpha$ and $K_\Omega \beta \cup \{\eta\} < F^\alpha(\xi)$.
2. $\alpha \leq \beta$ and $F^\beta(\eta) \leq K_\Omega \alpha$.

Proposition 4. Suppose that $\alpha < \varepsilon_{\Omega_1+1}$, a function $F : \text{Ord} \rightarrow \text{Ord}$ has a Σ_1 -definition in the Ω_1 -th stage L_{Ω_1} of the constructible hierarchy $(L_\alpha)_{\alpha \in \text{Ord}}$ and that $F(\xi) < \Omega_1$ for all $\xi < \Omega_1$. Then F^α also has a Σ_1 -definition in L_{Ω_1} and $F^\alpha(\xi) < \Omega_1$ holds for all $\xi < \Omega_1$.

Proof. By induction on $\alpha < \varepsilon_{\Omega_1+1}$. If $\alpha = 0$, then F^0 is a Σ_1 -function since so is F , and $F^0(\xi) = F(\xi) < \Omega_1$ for all $\xi < \Omega_1$. Suppose $\alpha > 0$. From elementary facts in generalised recursion theory, c.f. Barwise's book [3], careful readers will observe that F^α has a Σ_1 -definition in L_{Ω_1} since “ $\xi \in K_\Omega \alpha$ ” can be expressed by a Δ_0 -formula. To see that $F^\alpha(\xi) < \Omega_1$ for all $\xi < \Omega_1$ let us define a function $\psi : \omega \rightarrow \varepsilon_{\Omega_1}$ by

$$\begin{aligned} \psi(0) &= \min\{\gamma < \varepsilon_{\Omega_1+1} \mid \omega^\gamma = \gamma \text{ and } K_\Omega \alpha \cup \{\xi\} < \gamma\}, \\ \psi(m+1) &= \min\{\gamma < \varepsilon_{\Omega_1+1} \mid \omega^\gamma = \gamma, K_\Omega \alpha \cup \{\xi\} < \gamma \text{ and} \\ &\quad (\forall \eta < \psi(m))(\forall \beta < \alpha)[K_\Omega \beta < \psi(m) \Rightarrow F^\beta(\eta) < \gamma]\}. \end{aligned}$$

We can see that ψ is a Σ_1 -function in the same way as we see that F^α is so.

Claim. $\psi(m) < \Omega_1$ for all $m \in \omega$.

We show that $\psi(m) < \Omega_1$ holds by (side) induction on m . In the base case, $\psi(0) < \Omega_1$ holds since $K_\Omega \alpha \cup \{\xi\} < \Omega_1$ and Ω_1 is closed under the function [E]. Consider the induction step. Let $\eta < \psi(m)$. Then Side Induction Hypothesis implies $\eta < \psi(m) < \Omega_1$. Hence (Main) Induction Hypothesis enables us to deduce $F^\beta(\eta) < \Omega_1$ for all $\beta < \alpha$. Let us define a function $G : \{\beta < \alpha \mid K_\Omega \beta < \psi(m)\} \rightarrow \Omega_1$ by $\beta \mapsto F^\beta(\eta)$. One can see that G is a Σ_1 -function. On the other hand $\#\{\beta < \alpha \mid K_\Omega \beta < \psi(m)\} \leq \omega$ since $\psi(m) < \Omega_1$. Here we recall that Ω_1

denotes the least recursively regular ordinal ω_1^{CK} and hence L_{Ω_1} is closed under functions whose graphs are of Σ_1 in L_{Ω_1} . From these we have inequality

$$\psi(m+1) \leq \sup\{G(\beta) \mid \beta < \alpha \text{ and } K_{\Omega}\beta < \psi(m)\} < \Omega_1,$$

concluding the claim.

By the claim ψ is a Σ_1 -function in L_{Ω_1} from ω to Ω_1 . Hence $\sup_{m \in \omega} \psi(m) < \Omega_1$. Define an ordinal γ by $\gamma = \sup_{m \in \omega} \psi(m)$. Then $\omega^\gamma = \gamma$, $K_{\Omega}\alpha \cup \{\xi\} < \gamma$ and $K_{\Omega}\beta < \gamma \Rightarrow F^\beta(\eta) < \gamma$ for all $\eta < \xi$ and for all $\beta < \alpha$. This implies $F^\alpha(\xi) \leq \gamma < \Omega_1$. \square

Proposition 5. *For any $\alpha \in \text{Ord}$, for any $\eta, \xi < \Omega_1$ and for any ordinal function $F : \Omega_1 \rightarrow \Omega_1$, if $\eta < F^\alpha(\xi)$, then $F^\alpha(\eta) \leq F^\alpha(\xi)$.*

Proof. If $\eta \leq \xi$, then $F^\alpha(\eta) \leq F^\alpha(\xi)$ by the definition of $F^\alpha(\eta)$. Let us consider the case $\xi < \eta < F^\alpha(\xi)$. In this case $K_{\Omega}\alpha \cup \{\eta\} < F^\alpha(\xi)$ by the definition of $F^\alpha(\xi)$. Suppose that $\beta < \alpha$, $\gamma < F^\alpha(\xi)$ and $K_{\Omega}\beta < F^\alpha(\xi)$. Then $F^\beta(\gamma) < F^\alpha(\xi)$ again by the definition of $F^\alpha(\xi)$. By the minimality of $F^\alpha(\eta)$ we can conclude $F^\alpha(\eta) \leq F^\alpha(\xi)$. \square

Definition 6. *We define the value $[\alpha] \in \text{Ord}$ of an ordinal term $\alpha \in \mathcal{OT}(\mathcal{F})$ by recursion on the length of α .*

1. $[0] = 0$ and $[\Omega] = \Omega_1$.
2. $[\alpha + \beta] = [\alpha] + [\beta]$.
3. $[\varphi\alpha\beta] = [\varphi][\alpha][\beta]$, where $[\varphi]$ is the standard Veblen function, i.e.,

$$\left\{ \begin{array}{ll} [\varphi]0\beta = \omega^\beta, & \\ [\varphi](\alpha+1)0 = \sup\{([\varphi]\alpha)^n 0 \mid n \in \omega\}, & \\ [\varphi]\gamma 0 = \sup\{[\varphi]\alpha 0 \mid \alpha < \gamma\} & \text{if } \gamma \in \text{Lim}, \\ [\varphi](\alpha+1)(\beta+1) = \sup\{([\varphi]\alpha)^n([\varphi](\alpha+1)\beta+1 \mid n \in \omega\}, & \\ [\varphi]\gamma(\beta+1) = \sup\{[\varphi]\alpha([\varphi]\gamma\beta+1) \mid \alpha < \gamma\} & \text{if } \gamma \in \text{Lim}, \\ [\varphi]\alpha\gamma = \sup\{[\varphi]\alpha\beta \mid \beta < \gamma\} & \text{if } \gamma \in \text{Lim}. \end{array} \right.$$
4. $[\Omega^\alpha \cdot \xi] = \Omega_1^{[\alpha]} \cdot [\xi]$.
5. $[S(\alpha)] = [S]([\alpha])$, where $[S]$ denotes the ordinal successor $\alpha \mapsto \alpha + 1$. Clearly $\{[S](\xi) \mid \xi \in \Omega_1\} \subseteq \Omega_1$.
6. $[E(\alpha)] = [E]([\alpha])$, where the function $[E] : \text{Ord} \rightarrow \text{Ord}$ is defined by $[E](\alpha) = \min\{\xi \in \text{Ord} \mid \omega^\xi = \xi \text{ and } \alpha < \xi\}$. It is also clear that $\{[E](\xi) \mid \xi \in \Omega_1\} \subseteq \Omega_1$ holds.
7. $[F^\alpha(\xi)] = [F]^{[\alpha]}([\xi])$.

Definition 7. *For all $\alpha, \beta \in \mathcal{OT}(\mathcal{F})$, $\alpha < \beta$ if $[\alpha] < [\beta]$, and $\alpha = \beta$ if $[\alpha] = [\beta]$.*

We will identify each element $\alpha \in \mathcal{OT}(\mathcal{F})$ with its value $[\alpha] \in \text{Ord}$. Accordingly we will write $K_{\Omega}\alpha$ instead of $K_{\Omega}[\alpha]$ for $\alpha \in \mathcal{OT}(\mathcal{F})$. Further for a finite set $K \subseteq \text{Ord}$ we write $K_{\Omega}K$ to denote the finite set $\bigcup_{\xi \in K} K_{\Omega}\xi$. By this identification, \mathbb{H} is the set of *additively indecomposable* ordinals and SC is the set of *strongly critical* ordinals, i.e., $\text{SC} \subseteq \mathbb{H} \subseteq \text{Lim} \cup \{1\} \subseteq \text{Ord}$.

Corollary 8. $F^\alpha(\xi) < \Omega$ for any $F \in \mathcal{F}$ and $\xi < \Omega$.

Proof. Proof by induction over the build-up of $F \in \mathcal{F}$.

Corollary 9. 1. $K_\Omega 0 = K_\Omega \Omega = \emptyset$.

2. If $K_\Omega \alpha < \xi$ and $\xi \in \text{SC}$, then $K_\Omega \mathbf{S}(\alpha) < \xi$.

3. $K_\Omega \mathbf{E}(\alpha) = \{\mathbf{E}(\alpha)\}$ (since $\alpha < \Omega$).

4. If $K_\Omega \alpha \cup K_\Omega \beta < \xi$ and $\xi \in \text{SC}$, then $K_\Omega(\alpha + \beta) < \xi$.

5. $K_\Omega \varphi \alpha \beta = \{\varphi \alpha \beta\}$ (since $\alpha, \beta < \Omega$). Further, if $\alpha, \beta < \xi$ and $\xi \in \text{SC}$, then $\varphi \alpha \beta < \xi$.

6. $K_\Omega F^\alpha(\xi) = \{F^\alpha(\xi)\}$ (since $\xi < \Omega$).

By Corollary 8 each function symbol from \mathcal{F} defines a weakly increasing function $F : \Omega \rightarrow \Omega$ such that $\xi < F(\xi)$ holds for all $\xi \in \Omega$. In the rest of this section let F denote such a function. For a finite set $K \subseteq \text{Ord}$ we will use the notation $F[K](\xi)$ to abbreviate $F(\max(K \cup \{\xi\}))$.

Lemma 10. Let $K \subseteq \text{Ord}$ be a finite set such that $K < \Omega$. Then $(F[K])^\alpha(\xi) \leq F^\alpha[K](\xi)$ for all $\xi < \Omega$.

Proof. By induction on α . For the base case $(F[K])^0(\xi) = F[K](\xi) = F^0[K](\xi)$. Suppose $\alpha > 0$. Then

$$K_\Omega \alpha \cup \{\xi\} < F^\alpha(\xi) \leq F^\alpha[K](\xi). \quad (1)$$

Assume that $\eta < F^\alpha[K](\xi)$, $\beta < \alpha$ and $K_\Omega \beta < F^\alpha[K](\xi)$. Then $\eta < \Omega$, and hence $(F[K])^\beta(\eta) \leq F^\beta[K](\eta)$ by IH. Hence

$$\begin{aligned} (F[K])^\beta(\eta) &\leq F^\beta[K](\eta), \\ &< F^\alpha[K](\eta) \quad \text{since } K_\Omega K < F^\alpha[K](\eta). \end{aligned} \quad (2)$$

By conditions (1) and (2) we conclude $(F[K])^\alpha(\xi) \leq F^\alpha[K](\xi)$. \square

Lemma 11. $(F^\alpha)^\beta(\xi) \leq F^{\alpha+\beta}(\xi)$ for all $\xi < \Omega$.

Proof. By induction on β . For the base case $(F^\alpha)^0(\xi) = F^\alpha(\xi) = F^{\alpha+0}(\xi)$. Suppose $\beta > 0$. Then

$$K_\Omega \beta \cup \{\xi\} < F^\beta(\xi) \leq F^{\alpha+\beta}(\xi). \quad (3)$$

Assume that $\eta < F^{\alpha+\beta}(\xi)$, $\beta' < \beta$ and $K_\Omega \beta' < F^{\alpha+\beta}(\xi)$. Then $\eta < \Omega$, and hence $(F^\alpha)^{\beta'}(\eta) \leq F^{\alpha+\beta'}(\eta)$ by IH. Hence

$$(F^\alpha)^{\beta'}(\eta) \leq F^{\alpha+\beta'}(\eta) < F^{\alpha+\beta}(\xi). \quad (4)$$

By conditions (3) and (4) we can conclude $(F^\alpha)^\beta(\xi) \leq F^{\alpha+\beta}(\xi)$. \square

4 An infinitary proof system \mathbf{ID}_1^∞

This section introduces the main definition of this paper. We introduce a new infinitary proof system \mathbf{ID}_1^∞ to which the new ordinal notation system is connected and into which every (finite) proof in \mathbf{ID}_1 can be embedded in good order. For each positive operator form \mathcal{A} and for each ordinal term $\alpha \in (\mathcal{OT}(\mathcal{F}) \upharpoonright \Omega) \cup \{\Omega\}$ let $P_{\mathcal{A}}^{<\alpha}$ be a new unary predicate symbol. Let us define an infinitary language \mathcal{L}^* of \mathbf{ID}_1^∞ by $\mathcal{L}^* = \mathcal{L}_{\text{PA}} \cup \{\neq, \not\leq\} \cup \{P_{\mathcal{A}}^{<\alpha}, \neg P_{\mathcal{A}}^{<\alpha} \mid \alpha \in (\mathcal{OT}(\mathcal{F}) \upharpoonright \Omega) \cup \{\Omega\} \text{ and } \mathcal{A} \text{ is a positive operator form}\}$. Let us write $P_{\mathcal{A}}^{<\Omega}$ to denote $P_{\mathcal{A}}$ to have the inclusion $\mathcal{L}_{\text{ID}_1} \subseteq \mathcal{L}^*$. We write $\mathcal{T}(\mathcal{L}^*)$ to denote the set of closed \mathcal{L}^* -terms. Specifically, the language \mathcal{L}^* contains complementary predicate symbol $\neg P$ for each predicate symbol $P \in \mathcal{L}^*$. We note that the negation \neg nor the implication \rightarrow is not included as a logical symbol. The negation $\neg A$ is defined via de Morgan's law by $\neg(\neg P(t)) \equiv P(t)$ for an atomic formula $P(t)$, $\neg(A \wedge B) \equiv \neg A \vee \neg B$, $\neg(A \vee B) \equiv \neg A \wedge \neg B$, $\neg \forall x A \equiv \exists x \neg A$ and $\neg \exists x A \equiv \forall x \neg A$. The implication $A \rightarrow B$ is defined by $\neg A \vee B$. We start with technical definitions. We will write $P_{\mathcal{A}}^{<\alpha} t$ and $\neg P_{\mathcal{A}}^{<\alpha} t$ respectively for $P_{\mathcal{A}}^{<\alpha}(t)$ and $\neg P_{\mathcal{A}}^{<\alpha}(t)$.

Definition 12 (Complexity measures of \mathcal{L}^* -formulas).

1. The length $\text{lh}(A)$ of an \mathcal{L}^* -formula A is the number of the symbols $P_{\mathcal{A}}^{<\alpha}$, $\neg P_{\mathcal{A}}^{<\alpha}$, \vee , \wedge , \exists and \forall occurring in A .
2. The rank $\text{rk}(A)$ of an \mathcal{L}^* -formula A .
 - (a) $\text{rk}(P_{\mathcal{A}}^{<\alpha} t) := \text{rk}(\neg P_{\mathcal{A}}^{<\alpha} t) := \omega \cdot \alpha$.
 - (b) $\text{rk}(A) := 0$ if A is an $\mathcal{L}_{\text{ID}_1}$ -literal.
 - (c) $\text{rk}(A \wedge B) := \text{rk}(A \vee B) := \max\{\text{rk}(A), \text{rk}(B)\} + 1$.
 - (d) $\text{rk}(\forall x A) := \text{rk}(\exists x A) := \text{rk}(A) + 1$.
3. The set $\mathbf{k}^\Pi(A)$ of Π -coefficients of an \mathcal{L}^* -formula A .
 - (a) $\mathbf{k}^\Pi(P_{\mathcal{A}}^{<\alpha} t) := \{0\}$, $\mathbf{k}^\Pi(\neg P_{\mathcal{A}}^{<\alpha} t) := \{0, \alpha\}$.
 - (b) $\mathbf{k}^\Pi(A) := \{0\}$ if A is an $\mathcal{L}_{\text{ID}_1}$ -literal.
 - (c) $\mathbf{k}^\Pi(A \wedge B) := \mathbf{k}^\Pi(A \vee B) := \mathbf{k}^\Pi(A) \cup \mathbf{k}^\Pi(B)$.
 - (d) $\mathbf{k}^\Pi(\forall x A) := \mathbf{k}^\Pi(\exists x A) := \mathbf{k}^\Pi(A)$.
4. The set $\mathbf{k}^\Sigma(A)$ of Σ -coefficients of an \mathcal{L}^* -formula A .
 $\mathbf{k}^\Sigma(A) := \mathbf{k}^\Pi(\neg A)$.
5. The set $\mathbf{k}(A)$ of all the coefficients of an \mathcal{L}^* -formula A .
 $\mathbf{k}(A) := \mathbf{k}^\Pi(A) \cup \mathbf{k}^\Sigma(A)$.
6. The set $\mathbf{k}_\Omega^\Pi(A)$ of Π -coefficients of an \mathcal{L}^* -formula A less than Ω .
 $\mathbf{k}_\Omega^\Pi(A) := \mathbf{k}^\Pi(A) \upharpoonright \Omega$.
 The set $\mathbf{k}_\Omega^\Sigma(A)$ and $\mathbf{k}_\Omega(A)$ are defined accordingly.

By definition $\text{rk}(A) = \text{rk}(\neg A)$, $\mathbf{k}(A) = \mathbf{k}(\neg A)$ and $\mathbf{k}_\Omega(A) = \mathbf{k}_\Omega(\neg A)$.

Definition 13 (Complexity measures of \mathcal{L}^* -terms).

1. The value $\text{val}(t)$ of a term $t \in \mathcal{T}(\mathcal{L}_{\text{ID}_1}) = \mathcal{T}(\mathcal{L}_{\text{PA}})$ is the value of the closed term t in the standard model \mathbb{N} of the Peano arithmetic PA.

2. A complexity measure $\text{ord} : \mathcal{T}(\mathcal{L}^*) \rightarrow (\mathcal{OT}(\mathcal{F}) \upharpoonright \Omega) \cup \{\Omega\}$ is defined by

$$\begin{cases} \text{ord}(t) := 0 & \text{if } t \in \mathcal{T}(\mathcal{L}_{\text{ID}_1}), \\ \text{ord}(\alpha) := \xi & \text{if } \alpha \in \mathcal{OT}(\mathcal{F}). \end{cases}$$
3. The norm $N(\alpha)$ of $\alpha \in \mathcal{OT}(\mathcal{F})$.
 - (a) $N(0) = 0$ and $N(\Omega) = 1$.
 - (b) $N(\mathbf{S}(\alpha)) = N(\alpha) + 1$.
 - (c) $N(\mathbf{E}(\alpha)) = N(\alpha) + 1$.
 - (d) $N(\alpha + \beta) = N(\alpha) + N(\beta)$.
 - (e) $N(\varphi\alpha\beta) = N(\alpha) + N(\beta) + 1$,
 - (f) $N(\Omega^\alpha \cdot \xi) = N(\alpha) + N(\xi) + 1$.
 - (g) $N(F^\alpha(\xi)) = N(F(\xi)) + N(\alpha)$.
 The norm is extended to a complexity measure $N : \mathcal{T}(\mathcal{L}^*) \rightarrow \mathbb{N}$ by

$$\begin{cases} N(t) := \text{val}(t) & \text{if } t \in \mathcal{T}(\mathcal{L}_{\text{ID}_1}), \\ N(\alpha) := N(\alpha) & \text{if } \alpha \in \mathcal{OT}(\mathcal{F}). \end{cases}$$

By definition $N(\omega^\alpha) = N(\varphi 0 \alpha) = N(\alpha) + 1$ and $N(m) = N(\omega^0 \cdot m) = m$ for any $m < \omega$. This seems to be a good point to explain why we contain the constant Ω in $\mathcal{OT}(\mathcal{F})$. Having that $N(\Omega) = 1$ makes some technicality easier.

Definition 14. We define a relation \simeq between \mathcal{L}^* -sentences and (infinitary) propositional \mathcal{L}^* -sentences.

1. $\neg P_{\mathcal{A}}^{<\alpha} t \simeq \bigwedge_{\xi \in \mathcal{OT}(\mathcal{F}) \upharpoonright \alpha} \neg \mathcal{A}(P_{\mathcal{A}}^{<\xi} t)$ and $P_{\mathcal{A}}^{<\alpha} t \simeq \bigvee_{\xi \in \mathcal{OT}(\mathcal{F}) \upharpoonright \alpha} \mathcal{A}(P_{\mathcal{A}}^{<\xi} t)$.
2. $A \wedge B \simeq \bigwedge_{\iota \in \{\underline{0}, \underline{1}\}} A_\iota$ and $A \vee B \simeq \bigvee_{\iota \in \{\underline{0}, \underline{1}\}} A_\iota$ where $A_{\underline{0}} \equiv A$ and $A_{\underline{1}} \equiv B$.
3. $\forall x A(x) \simeq \bigwedge_{t \in \mathcal{T}(\mathcal{L}_{\text{ID}_1})} A(t)$ and $\exists x A(x) \simeq \bigvee_{t \in \mathcal{T}(\mathcal{L}_{\text{ID}_1})} A(t)$.

We call an \mathcal{L}^* -sentence A a \bigwedge -type (conjunctive type) if $A \simeq \bigwedge_{\iota \in J} A_\iota$ for some A_ι , and a \bigvee -type (disjunctive type) if $A \simeq \bigvee_{\iota \in J} A_\iota$ for some A_ι . For the sake of simplicity we will write $\bigwedge_{\xi < \alpha} A_\xi$ instead of $\bigwedge_{\xi \in \mathcal{OT}(\mathcal{F}) \upharpoonright \alpha} A_\xi$ and write $\bigvee_{\xi < \alpha} A_\xi$ accordingly.

- Lemma 15.**
1. If either $A \simeq \bigwedge_{\iota \in J} A_\iota$ or $A \simeq \bigvee_{\iota \in J} A_\iota$, then for all $\iota \in J$, $\mathbf{k}^\Pi(A_\iota) \subseteq \mathbf{k}^\Pi(A) \cup \{\text{ord}(\iota)\}$ and $\mathbf{k}^\Sigma(A_\iota) \subseteq \mathbf{k}^\Sigma(A) \cup \{\text{ord}(\iota)\}$.
 2. For any $\alpha \in \mathcal{OT}(\mathcal{F})$, if $A \simeq \bigwedge_{\xi < \alpha} A_\xi$, then $(\exists \sigma \in \mathbf{k}^\Pi(A))(\forall \xi < \alpha)[\xi \leq \sigma]$.
 3. For any \mathcal{L}^* -sentence A , $\text{rk}(A) = \omega \cdot \max \mathbf{k}(A) + n$ for some $n \leq \text{lh}(A)$.
 4. If $\text{rk}(A) = \Omega$, then either $A \equiv P_{\mathcal{A}}^{<\Omega} t$ or $A \equiv \neg P_{\mathcal{A}}^{<\Omega} t$.
 5. If either $A \simeq \bigwedge_{\iota \in J} A_\iota$ or $A \simeq \bigvee_{\iota \in J} A_\iota$, then for all $\iota \in J$, $N(\text{rk}(A_\iota)) \leq \max\{N(\text{rk}(A)), 2 \cdot N(\iota)\}$.

Throughout this section we use the symbol F to denote a weakly increasing ordinal function $F : \Omega \rightarrow \Omega$ and the symbol f to denote a number-theoretic function $f : \mathbb{N} \rightarrow \mathbb{N}$ that enjoys the following conditions.

- (f.1) f is a strictly increasing function such that $2m + 1 \leq f(m)$ for all m .
Hence, in particular, $n + f(m) \leq f(n + m)$ for all m and n .
- (f.2) $2 \cdot f(m) \leq f(f(m))$ for all m .

We will use the notation $f[n](m)$ to abbreviate $f(n+m)$. It is easy to see that if the conditions (f.1) and (f.2) hold, then for a fixed n the conditions (f[n].1) and (f[n].2) also hold.

Definition 16. Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be a number-theoretic function. Then a function $f^\alpha : \mathbb{N} \rightarrow \mathbb{N}$ is defined by transfinite recursion on $\alpha \in \mathcal{OT}(\mathcal{F})$ by

$$\begin{aligned} f^0(m) &= f(m), \\ f^\alpha(m) &= \max\{f^\beta(f^\beta(m)) \mid \beta < \alpha \text{ and } N(\beta) \leq f[N(\alpha)](m)\} \quad \text{if } 0 < \alpha. \end{aligned}$$

Corollary 17. 1. If f is strictly increasing, then so is f^α for any $\alpha \in \mathcal{OT}(\mathcal{F})$.
2. If $\beta < \alpha$ and $N(\beta) \leq f[N(\alpha)](m)$, then $f^\beta(m) < f^\alpha(m)$.
3. $f^\alpha(f^\alpha(m)) \leq f^{\alpha+1}(m)$.

We note that the function f^α is not a recursive function in general even if f is recursive since the ordinal notation system $\langle \mathcal{OT}(\mathcal{F}), < \rangle$ is not a recursive system.

Example 18. The following are examples of f^α in case that $\alpha \leq \omega$ and f is the successor function $s : m \mapsto m+1$. Let us recall that $N(n) = N(\omega^0 \cdot n) = n$ for all $n < \omega$.

1. $s^1(m) = s^0(s^0(m)) = m+2$.
2. $s^2(m) = s^1(s^1(m)) = m+4$.
3. $s^n(m) = m+2^n$. ($n < \omega$)
4. $s^\omega(m) = m+2^{m+3}$.

Let us see that $N(\omega) = 1$ and hence $s[N(\omega)](m) = s(1+m) = m+2$. Hence $s^\omega(m) = f^{m+2}(f^{m+2}(m)) = m+2^{m+2}+2^{m+2} = m+2^{m+3}$.

Lemma 19. Let $\alpha \in \mathcal{OT}(\mathcal{F})$ and $F \in \mathcal{F}$. Then $N(\alpha) \leq f^{F^\alpha(0)}(0)$.

Proof. By induction over the term-construction of $\alpha \in \mathcal{OT}(\mathcal{F})$. For the base case $N(0) = 0 \leq f(0) \leq f^{F^0(0)}(0)$ and $N(\Omega) = 1 \leq f(0) \leq f^{F^\Omega(0)}(0)$. For the induction step, we only consider the case that $\alpha = F^{\alpha_0}(\xi)$ for some $\alpha_0 \neq 0$ and for some $\xi < \Omega$. The remaining cases can be treated in similar ways. In this case $F^{\alpha_0}(0) < F^\alpha(0)$ holds since $F^{\alpha_0}(0) \leq \{F^{\alpha_0}(\xi)\} = K_\Omega F^{\alpha_0}(\xi) < F^{F^{\alpha_0}(\xi)}(0) = F^\alpha(0)$. It is easy to see that $F^{F(\xi)}(0) < F^\alpha(0)$ holds. By definition $N(\alpha) = N(F(\xi)) + N(\alpha_0)$. By IH $N(F(\xi)) \leq f^{F^{F(\xi)}(0)}(0)$ and $N(\alpha_0) \leq f^{F^{\alpha_0}(0)}(0)$. Hence

$$\begin{aligned} N(\alpha) &\leq f^{F^{F(\xi)}(0)}(0) + f^{F^{\alpha_0}(0)}(0), \\ &\leq f^{F^{\alpha_0}(0)}(f^{F^{F(\xi)}(0)}(0)) \text{ since } m + f^{\omega^{\alpha_0}}(0) \leq f^{\omega^{\alpha_0}}(m) \text{ for all } m, \\ &\leq f^{F^{\alpha_0}(0) + F^{F(\xi)}(0)}(f^{F^{\alpha_0}(0) + F^{F(\xi)}(0)}(0)) \\ &\leq f^{F^\alpha(0)}(0). \end{aligned}$$

To see that the last inequality is true, we can check $F^{\alpha_0}(0) + F^{F(\xi)}(0) < F^\alpha(0)$ and $N(F^{\alpha_0}(0) + F^{F(\xi)}(0)) \leq 2 \cdot N(F^\alpha(0)) \leq f[N(F^\alpha(0))](0)$ from an assumption that $2m \leq f(m)$. \square

Lemma 20. Let $\{\alpha, \beta\} \subseteq \mathcal{OT}(\mathcal{F}) \upharpoonright \Omega$ and $F \in \mathcal{F}$. Then, for all m , $(f^\alpha)^\beta(m) \leq f^{F^{\Omega \cdot \alpha + \beta}(0)}(m)$.

Proof. If $\alpha = 0$, then $(f^\alpha)^\beta(m) = f^\beta(m) \leq f^{F^{\Omega \cdot 0 + \beta}(0)}(m)$. Suppose $\alpha \neq 0$. Then we show the assertion by induction on β . If $\beta = 0$, then $(f^\alpha)^\beta(m) = f^\alpha(m) \leq f^{F^{\Omega \cdot \alpha}(0)}(m)$. Suppose $\beta > 0$. Then there exists $\gamma < \beta$ such that $N(\gamma) \leq f^\alpha[N(\beta)](m)$ and $(f^\alpha)^\beta(m) = (f^\alpha)^\gamma((f^\alpha)^\gamma(m))$. By IH

$$(f^\alpha)^\gamma((f^\alpha)^\gamma(m)) \leq f^{F^{\Omega \cdot \alpha + \gamma}(0)}(f^{F^{\Omega \cdot \alpha + \gamma}(0)}(m)). \quad (5)$$

On the other hand $N(\beta) \leq f^{F^\beta(0)}(0)$ by Lemma 19. Hence

$$\begin{aligned} N(\gamma) &\leq f^\alpha(f^{F^\beta(0)}(m)) \quad \text{since } m \leq f(m) \leq f^{F^\beta(0)}(m), \\ &\leq f^{F^{\Omega \cdot \alpha}(0) + F^\beta(0)}(f^{F^{\Omega \cdot \alpha}(0) + F^\beta(0)}(m)) \\ &\leq f^{F^{\Omega \cdot \alpha + \beta}(0)}(m). \end{aligned}$$

The second inequality holds since $\{\alpha, F^\beta(0)\} = K_\Omega \alpha \cup \{F^\beta(0)\} < F^{\Omega \cdot \alpha}(0) + F^\beta(0)$. This implies that

$$\begin{aligned} N(F^{\Omega \cdot \alpha + \gamma}(0)) &\leq N(F(0)) + N(\alpha) + 1 + f^{F^{\Omega \cdot \alpha + \beta}(0)}(m) \\ &\leq f[N(F^{\Omega \cdot \alpha + \beta}(0))](f^{F^{\Omega \cdot \alpha + \beta}(0)}(m)). \end{aligned} \quad (6)$$

Further $F^{\Omega \cdot \alpha + \gamma}(0) < F^{\Omega \cdot \alpha + \beta}(0)$ holds since $K_\Omega \gamma = \{\gamma\} < \beta$. This together with the inequality (6) yields that

$$\begin{aligned} (f^\alpha)^\beta(m) &\leq f^{F^{\Omega \cdot \alpha + \gamma}(0)}(f^{F^{\Omega \cdot \alpha + \gamma}(0)}(m)) \quad \text{by (5),} \\ &\leq f^{F^{\Omega \cdot \alpha + \beta}(0)}(m). \end{aligned}$$

□

Lemma 21. 1. $f^\alpha[n](m) \leq (f[n])^\alpha(m)$.

2. If $n \leq m$, then $(f[n])^\alpha(m) \leq f^\alpha[f^\alpha(f(m))](f(m))$.

We write $f[n][m]$ to abbreviate $(f[n])(m)$ and $f[n]^\alpha$ to abbreviate $(f[n])^\alpha$.

Proof. PROPERTY 1. By induction on α . For the base case $f^0[n](m) = f[n](m) = f[n]^0(m)$. For the induction step, assume $\alpha > 0$. Then there exists $\beta < \alpha$ such that $N(\beta) \leq f[N(\alpha)][n](m)$ and $f^\alpha[n](m) = f^\beta(f^\beta[n](m))$. Hence

$$\begin{aligned} f^\alpha[n](m) &\leq f^\beta(f[n]^\beta(m)) \quad \text{by IH,} \\ &\leq f[n]^\beta(f[n]^\beta(m)) \\ &\leq f[n]^\alpha(m). \end{aligned}$$

The last inequality holds since $N(\beta) \leq f[N(\alpha)][n](m) = f[n][N(\alpha)](m)$.

PROPERTY 2. We show that $f[n]^\alpha(f(m)) \leq f^\alpha[f^\alpha(f(m))](f(m))$ holds for all $m \geq n$ by induction on α . Let $n \leq m$. For the base case $f[n]^0(m) \leq f[n](m) \leq$

$f(m+m) \leq f(f^0(f(m)) + f(m)) = f^0[f^0(f(m))](f(m))$. For the induction step, assume $\alpha > 0$. Then there exists $\beta < \alpha$ such that $N(\beta) \leq f[n][N[\alpha]](m)$ and $f[n]^\alpha(m) = f[n]^\beta(f[n]^\beta(m))$. Let us observe that

$$\begin{aligned} N(\beta) &= f(n + N(\alpha) + m) \leq f(N(\alpha) + 2m) \quad \text{since } n \leq m, \\ &\leq f(N(\alpha) + f(m)) \quad \text{from (f.1)}. \end{aligned} \quad (7)$$

We can see that $f[n]^\alpha(f(m)) \leq f^\alpha[f^\alpha(f(m))](f(m))$ holds as follows.

$$\begin{aligned} f[n]^\alpha(m) &\leq f^\beta(f^\beta(f(m)) + f^\beta(f^\beta(f(m)) + f(m))) \quad \text{by IH,} \\ &\leq f^\beta(f^\beta(2 \cdot f^\beta(f(m)) + f(m))) \quad \text{by (f}^\beta\text{.1),} \\ &\leq f^\beta(f^\beta(f^\beta(f^\beta(f(m)))) + f(m)) \quad \text{by (f}^\beta\text{.2),} \\ &\leq f^\beta(f^\beta(f^\alpha(f(m))) + f(m)) \quad \text{by (7),} \\ &\leq f^\alpha(f^\alpha(f(m)) + f(m)) = f^\alpha[f^\alpha(f(m))](f(m)). \end{aligned}$$

The last inequality holds since $N(\beta) \leq f(N(\alpha) + f^\alpha(f(m)) + f(m))$. \square

Corollary 22. *If $n \leq m$, then $(f[n])^\alpha(m) \leq f^{\alpha+2}(m)$.*

Proof. By Lemma 21.2, $f[n]^\alpha(m) \leq f^\alpha(f^\alpha(f(m)) + f(m)) \leq f^\alpha(f^\alpha(2 \cdot f(m))) \leq f^{\alpha+1}(f^0(f^0(m))) \leq f^{\alpha+1}(f^{\alpha+1}(m)) \leq f^{\alpha+2}(m)$. \square

We define a relation $f, F \vdash_\rho^\alpha \Gamma$ for a quintuple $(f, F, \alpha, \rho, \Gamma)$ where $\alpha < \varepsilon_{\Omega+1}$, $\rho < \Omega \cdot \omega$ and Γ is a sequent of \mathcal{L}^* -sentences. In this paper a “sequent” means a finite set of formulas. We write Γ, A or A, Γ to denote $\Gamma \cup \{A\}$. Let us recall that for a finite set $K \subseteq \text{Ord}$, $F[K](\xi)$ denotes $F(\max(K \cup \{\xi\}))$. We will write $F[\mu](\xi)$ to denote $F[\{\mu\}](\xi)$. We write TRUE_0 to denote the set $\{A \mid A \text{ is an } \mathcal{L}_{\text{PA}}\text{-literal true in the standard model } \mathbb{N} \text{ of PA}\}$.

Definition 23. $f, F \vdash_\rho^\alpha \Gamma$ if

$$\max\{N(F(0)), N(\alpha)\} \leq f(0), \quad K_\Omega \alpha < F(0), \quad (\text{HYP}(f; F; \alpha))$$

and one of the following holds.

- (Ax1) $\exists A(x)$: an $\mathcal{L}_{\text{ID}_1}$ -literal, $\exists s, t \in \mathcal{T}(\mathcal{L}_{\text{ID}_1})$ s.t. $\text{FV}(A) = \{x\}$, $\text{val}(s) = \text{val}(t)$ and $\{\neg A(s), A(t)\} \subseteq \Gamma$.
- (Ax2) $\Gamma \cap \text{TRUE}_0 \neq \emptyset$.
- (V) $\exists A \simeq \bigvee_{\iota \in J} A_\iota \in \Gamma$, $\exists \alpha_0 < \alpha$, $\exists \iota_0 \in J$ s.t. $N(\iota_0) \leq f(0)$ $\text{ord}(\iota_0) < \min\{\alpha, F(0)\}$, and $f, F \vdash_{\rho}^{\alpha_0} \Gamma, A_{\iota_0}$.
- (\(\wedge\)) $\exists A \simeq \bigwedge_{\iota \in J} A_\iota \in \Gamma$ s.t. $N(\max k_\Omega^\Pi(A)) \leq f(0)$, $k_\Omega^\Pi(A) < F(0)$ and $(\forall \iota \in J) [f[N(\iota)], F[\text{ord}(\iota)] \vdash_{\rho}^{\alpha_\iota} \Gamma, A_\iota]$.
- (Cl $_\Omega$) $\exists t \in \mathcal{T}(\mathcal{L}_{\text{ID}_1})$, $\exists \alpha_0 < \alpha$ s.t. $P_A^{\leq \Omega} t \in \Gamma$, $\Omega < \alpha$ and $f, F \vdash_{\rho}^{\alpha_0} \Gamma, A(P_A^{\leq \Omega} t)$.
- (Cut) $\exists C$: an \mathcal{L}^* -sentence of \bigvee -type, $\exists \alpha_0 < \alpha$ s.t. $\max\{\text{lh}(C), N(\max k_\Omega^\Pi(C)), N(\max k_\Omega^\Sigma(C))\} \leq f(0)$, $k_\Omega(C) < F(0)$, $\text{rk}(C) < \rho$, $f, F \vdash_{\rho}^{\alpha_0} \Gamma, C$, and $f, F \vdash_{\rho}^{\alpha_0} \Gamma, \neg C$.

We will call the pair (f, F) operators controlling the derivation that forms $f, F \vdash_\rho^\alpha \Gamma$.

In the sequel we always assume that the operator F enjoys the following condition $(\text{HYP}(F))$:

$$\eta < F(\xi) \Rightarrow F(\eta) \leq F(\xi) \quad \text{for any ordinals } \xi, \eta < \Omega. \quad (\text{HYP}(F))$$

We note that the hypothesis $(\text{HYP}(F))$ reflects the fact stated in Proposition 5. It is not difficult to see that if the condition $(\text{HYP}(F))$ holds, then the condition $(\text{HYP}(F[K]))$ also holds for any finite set $K < \Omega$.

Lemma 24 (Inversion). *Assume that $A \simeq \bigwedge_{\iota \in J} A_\iota$. If $f, F \vdash_\rho^\alpha \Gamma, A$, then $f[N(\iota)], F[\text{ord}(\iota)] \vdash_\rho^\alpha \Gamma, A_\iota$ for all $\iota \in J$.*

Proof. By induction on α . Let $\iota \in J$. Then we can check that the condition $\text{HYP}(f[N(\iota)]; F[\text{ord}(\iota)]; \alpha)$ holds. In particular, by the hypothesis $\text{HYP}(f; F; \alpha)$ we have $N(F[\text{ord}(\iota)]) = N(\iota) + N(F(0)) \leq N(\iota) + f(0) \leq f[N(\iota)](0)$. Now the assertion is a straightforward consequence of IH. \square

We write $f \circ g$ to denote the result $m \mapsto f(g(m))$ of composing f and g .

Lemma 25 (Cut-reduction). *Assume that $C \simeq \bigvee_{\iota \in J} C_\iota$, $\text{rk}(C) = \rho \neq \Omega$, $\max\{\text{lh}(C), N(\max \mathbf{k}_\Omega^H(C)), N(\max \mathbf{k}_\Omega^\Sigma(C))\} \leq f(g(0))$, and that $\mathbf{k}_\Omega(C) < F(0)$. If $f, F \vdash_\rho^\alpha \Gamma, \neg C$ and $g, F \vdash_\rho^\beta \Gamma, C$, then $f \circ g, F \vdash_\rho^{\alpha+\beta} \Gamma$.*

Proof. By induction on β .

CASE. C is not the principal formula of the last rule (\mathcal{J}) that forms $g, F \vdash_\rho^\beta \Gamma, C$: We only consider the case that (\mathcal{J}) is (\bigwedge) . The other cases can be treated similarly. Let us suppose that the sequent Γ contains a formula $\bigwedge_{\iota \in J} A_\iota$ and the inference rule (\mathcal{J}) has the premises $g[N(\iota)], F[\text{ord}(\iota)] \vdash_\rho^{\beta_\iota} \Gamma, A_\iota, C$ ($\iota \in J$) for some $\beta_\iota < \beta$. Then, since $f \circ (g[N(\iota)](0)) = (f \circ g)[N(\iota)](0)$ and $F(0) \leq F[\text{ord}(\iota)](0)$ for all $\iota \in J$, IH yields the sequent

$$(f \circ g)[N(\iota)], F[\text{ord}(\iota)] \vdash_\rho^{\alpha+\beta_\iota} \Gamma, A_\iota$$

for all $\iota \in J$. Hence another application of (\bigwedge) yields the sequent $f \circ g, F \vdash_\rho^{\alpha+\beta} \Gamma$.

CASE. C is the principal formula of the last rule (\mathcal{J}) : In this case (\mathcal{J}) should be (\bigvee) since $\text{rk}(C) \neq \Omega$. Let the premise be of the form $g, F \vdash_\rho^{\beta_0} \Gamma, C_{\iota_0}, C$ for some $\beta_0 < \beta$ and $\iota_0 \in J$ such that $N(\iota_0) \leq g(0)$ and $\text{ord}(\iota_0) < \min\{\beta, F(0)\}$. IH yields the sequent

$$f \circ g, F \vdash_\rho^{\alpha+\beta_0} \Gamma, C_{\iota_0}. \quad (8)$$

On the other hand, Inversion lemma yields the sequent $f[N(\iota_0)], F[\text{ord}(\iota_0)] \vdash_\rho^\alpha \Gamma, \neg C_{\iota_0}$. Let us observe the following. First, $f[N(\iota_0)](0) = f(N(\iota_0)) \leq f(g(0)) = (f \circ g)(0)$ since $N(\iota_0) \leq g(0)$. Secondly, $F[\text{ord}(\iota_0)](0) \leq F(0)$ by the hypothesis $\text{HYP}(F)$ since $\text{ord}(\iota_0) < F(0)$. Hence

$$f \circ g, F \vdash_\rho^{\alpha+\beta_0} \Gamma, \neg C_{\iota_0}. \quad (9)$$

We also observe that $N(\alpha + \beta) \leq N(\alpha) + N(\beta) \leq f(0) + g(0) \leq (f \circ g)(0)$. Further $K_\Omega(\alpha + \beta) < F(0)$ since $K_\Omega\alpha \cup K_\Omega\beta < F(0)$. Now by an application of (Cut) to the two sequents (8) and (9) we obtain $f \circ g, F \vdash_{\rho}^{\alpha+\beta} \Gamma$.

The other cases are similar. \square

For a sequent Γ we write $\mathbf{k}_\Omega^H(\Gamma)$ to denote the set $\bigcup_{B \in \Gamma} \mathbf{k}_\Omega^H(B)$.

Lemma 26. *Let $k < \omega$. If $f, F \vdash_{\Omega+k+2}^\alpha \Gamma$, then $f^{F^\alpha(0)+1}, F \vdash_{\Omega+k+1}^{\Omega^\alpha} \Gamma$.*

Proof. By induction on α . The argument splits into several cases depending on the last rule that forms $f, F \vdash_{\Omega+k+2}^\alpha \Gamma$. We only consider the following two critical cases. Let K denote the set $\mathbf{k}_\Omega^H(\Gamma)$.

CASE. The last rule is (Cut): In this case there are two premises $f, F \vdash_{\Omega+k+2}^{\alpha_0} \Gamma, C$ and $f, F \vdash_{\Omega+k+2}^{\alpha_0} \Gamma, \neg C$ with a cut formula C for some $\alpha_0 < \alpha$ such that $\text{rk}(C) < \Omega + k + 2$, $\max\{\text{lh}(C), N(\max \mathbf{k}_\Omega^H(C)), N(\max \mathbf{k}_\Omega^\Sigma(C))\} \leq f(0)$ and $\mathbf{k}_\Omega(C) < F(0)$. Let K_0 denote the set $\mathbf{k}_\Omega^H(\Gamma, \neg C)$. Then IH yields the two sequents

$$f^{F^{\alpha_0}[K_0](0)+1}, F \vdash_{\Omega+k+1}^{\Omega^{\alpha_0}} \Gamma, C, \quad f^{F^{\alpha_0}[K_0](0)+1}, F \vdash_{\Omega+k+1}^{\Omega^{\alpha_0}} \Gamma, \neg C.$$

Hence Cut-reduction lemma yields the sequent

$$f^{F^{\alpha_0}[K_0](0)+1} \circ f^{F^{\alpha_0}[K_0](0)+1}, F \vdash_{\Omega+k+1}^{\Omega^{\alpha_0} + \Omega^{\alpha_0}} \Gamma.$$

Clearly $\Omega^{\alpha_0} + \Omega^{\alpha_0} < \Omega^\alpha$. Further $N(\Omega^\alpha) = N(\alpha) + 1 \leq f^{F^\alpha[K](0)+1}(0)$ since $N(\alpha) \leq f(0) = f^0(0) < f^{F^\alpha(0)+1}(0)$. It remains to show that

$$(f^{F^{\alpha_0}[K_0](0)+1} \circ f^{F^{\alpha_0}[K_0](0)+1})(0) \leq f^{F^\alpha[K](0)+1}(0).$$

Let us see that $K_0 \subseteq K \cup \mathbf{k}_\Omega(C) < F^\alpha[K](0)$ since $\mathbf{k}_\Omega(C) < F(0)$. This implies $F^{\alpha_0}[K_0](0) < F^\alpha[K](0)$, and hence $F^{\alpha_0}[K_0](0) + 1 < F^\alpha[K](0)$. We can also see that

$$N(\max K_0) \leq \max\{N(\max K), N(\max \mathbf{k}_\Omega^\Sigma(C))\} \leq \max\{N(\max K), f(0)\}.$$

From this and the inequality $N(\alpha_0) \leq f(0)$ one can see that

$$\begin{aligned} N(F^{\alpha_0}[K_0](0) + 1) &\leq N(F[K_0](0)) + N(\alpha_0) + 1 \\ &\leq N(F[K](0)) + f(0) + f(0) + 1 \\ &\leq f(N(F^\alpha[K](0)) + f^{F^\alpha[K](0)}(0)). \end{aligned}$$

This allows us to conclude as follows.

$$\begin{aligned} &(f^{F^{\alpha_0}[K_0](0)+1} \circ f^{F^{\alpha_0}[K_0](0)+1})(0) \\ &\leq (f^{F^{\alpha_0}[K_0](0)+1} \circ f^{F^{\alpha_0}[K_0](0)+1})(f^{F^\alpha[K](0)}(0)) \\ &\leq f^{F^\alpha[K](0)}(f^{F^\alpha[K](0)}(0)) \\ &\leq f^{F^\alpha[K](0)+1}(0). \end{aligned}$$

CASE. The last rule is (\bigwedge): In this case there exists a formula $A \simeq \bigwedge_{\iota \in J} A_\iota \in \Gamma$ such that $N(\max \mathbf{k}_\Omega^H(A)) \leq f(0)$, $\mathbf{k}_\Omega^H(A) < F(0)$ and $\forall \iota \in J, \exists \alpha_\iota < \alpha$ s.t. $f[N(\iota)], F[\text{ord}(\iota)] \vdash_{\Omega+k+2}^{\alpha_\iota} \Gamma, A_\iota$. By IH $(f[N(\iota)])^{F[\text{ord}(\iota)]^{\alpha_\iota}(0)+1}, F[\text{ord}(\iota)] \vdash_{\Omega+k+1}^{\Omega^{\alpha_\iota}} \Gamma, A_\iota$ for all $\iota \in J$.

Claim. $(f[N(\iota)])^{F[\text{ord}(\iota)]^{\alpha_\iota}(0)+1}(0) \leq f^{F^\alpha(0)+1}[N(\iota)](0)$ for all $\iota \in J$.

Assuming the claim, $f^{F^\alpha(0)+1}[N(\iota)], F[\text{ord}(\iota)] \vdash_{\Omega+k+1}^{\Omega^{\alpha_\iota}} \Gamma, A_\iota$ for all $\iota \in J$ and hence an application of (\bigwedge) yields $f^{F^\alpha(0)+1}, F \vdash_{\Omega+k+1}^{\Omega^\alpha} \Gamma$. To show the claim fix $\iota \in J$ arbitrarily and let $n := N(\iota)$. Then Corollary 22 yields

$$f[n]^{F[\text{ord}(\iota)]^{\alpha_\iota}(0)+1}(0) \leq f^{F[\text{ord}(\iota)]^{\alpha_\iota}(0)+3}(n). \quad (10)$$

By Lemma 15.2, $\text{ord}(\iota) \leq k_\Omega^H(A)$ since $\text{ord}(\iota) < \Omega$. Hence $\text{ord}(\iota) < F(0)$ since $k_\Omega^H(A) < F(0)$. This together with the hypothesis $(\text{HYP}(F))$ yields $K_\Omega \alpha_\iota < F[\text{ord}(\iota)] \leq F(0) \leq F^\alpha(0)$. Further $F[\text{ord}(\iota)]^{\alpha_\iota}(0) \leq F^{\alpha_\iota}(\text{ord}(\iota))$ by Lemma 10. Hence $F[\text{ord}(\iota)]^{\alpha_\iota}(0) = F^{\alpha_\iota}(\text{ord}(\iota)) < F^\alpha(0)$ since $\text{ord}(\iota) < F(0) \leq F^\alpha(0)$. And hence

$$F[\text{ord}(\iota)]^{\alpha_\iota}(0) + 3 < F^\alpha(0). \quad (11)$$

As in Example 18 we can see that $2n + 3 \leq f^\omega(n) \leq f^{F^\alpha(0)}(n)$. Hence

$$\begin{aligned} & N(F[\text{ord}(\iota)]^{\alpha_\iota}(0) + 3) \\ &= N(F(0)) + N(\text{ord}(\iota)) + N(\alpha_\iota) + 3 \\ &\leq N(F^\alpha(0)) + n + f(n) + 3 \quad \text{since } N(\alpha_\iota) \leq f[N(\iota)](0) = f(n), \\ &\leq f(N(F^\alpha(0)) + 2n + 3) \quad \text{from the condition (f.1),} \\ &\leq f(N(F^\alpha(0)) + f^{F^\alpha(0)}(n)). \end{aligned} \quad (12)$$

The two conditions (11) and (12) allows us to deduce that

$$\begin{aligned} f^{F[\text{ord}(\iota)]^{\alpha_\iota}(0)+3}(n) &\leq f^{F[\text{ord}(\iota)]^{\alpha_\iota}(0)+3}(f^{F^\alpha(0)}(n)) \\ &\leq f^{F^\alpha(0)}(f^{F^\alpha(0)}(n)) \\ &\leq f^{F^{\alpha+1}(0)}(n) = f^{F^{\alpha+1}(0)}[n](0). \end{aligned} \quad (13)$$

Combining the two inequality (10) and (13) enables us to conclude the claim, and hence completes this case. \square

Lemma 27 (Predicative Cut-elimination). *Assume $\{\alpha, \beta, \gamma\} < \Omega$, $N(\alpha) \leq f^\gamma(0)$ and $K_\Omega \alpha < F(0)$. If $f^\gamma, F \vdash_{\rho+\omega^\alpha}^\beta \Gamma$, then $f^{F^{\Omega \cdot \alpha + \gamma + \beta}(0)+1}, F \vdash_\rho^{\varphi \alpha \beta} \Gamma$.*

Proof. By main induction on α and side induction on β . Let us start with observing the following. First $N(\varphi \alpha \beta) = N(\alpha) + N(\beta) + 1 \leq f^\gamma(0) + f^\gamma(0) + 1 \leq f^\gamma(f^\gamma(0)) + 1 \leq f^{F^{\Omega \cdot \alpha + \gamma + \beta}(0)+1}(0)$. Secondly $K_\Omega \varphi \alpha \beta = \{\varphi \alpha \beta\} < F(0)$ since $K_\Omega \alpha \cup K_\Omega \beta < F(0)$.

CASE. The last rule is (\bigwedge) : In this case there exists a formula $A \simeq \bigwedge_{\iota \in J} A_\iota \in \Gamma$ and for all $\iota \in J$ there exists $\beta_\iota < \beta$ such that $f^\gamma[N(\iota)], F[\text{ord}(\iota)] \vdash_{\rho+\omega^\alpha}^{\beta_\iota} \Gamma, A_\iota$. We observe that $N(\alpha) \leq f^\gamma(0) \leq f[N(\iota)]^\gamma(0)$ and $K_\Omega \alpha < F(0) \leq F[\text{ord}(\iota)](0)$. Hence Side Induction Hypothesis yields that for all $\iota \in J$

$$f[N(\iota)]^{F^{\Omega \cdot \alpha + \gamma + \beta_\iota}(0)+1}, F[\text{ord}(\iota)] \vdash_\rho^{\varphi \alpha \beta_\iota} \Gamma, A_\iota. \quad (14)$$

Let $m := N(\iota)$. Then $f[m]^{F^{\Omega \cdot \alpha + \gamma + \beta_\iota}(0)+1}(0) \leq f^{F^{\Omega \cdot \alpha + \gamma + \beta_\iota}(0)+3}[m](0)$ from Corollary 22. Also it holds that $F^{\Omega \cdot \alpha + \gamma + \beta_\iota}(0) < F^{\Omega \cdot \alpha + \gamma + \beta}(0)$ for all $\iota \in J$ since $K_\Omega \beta_\iota < F[\text{ord}(\iota)](0) \leq F(0)$. Further

$$\begin{aligned}
N(F^{\Omega \cdot \alpha + \gamma + \beta_\iota}(0) + 3) &= N(F(0)) + N(\alpha) + N(\gamma) + N(\beta_\iota) + 4 \\
&\leq 2 \cdot f^\gamma(0) + f^{F^\gamma(0)}(0) + f^\gamma[m](0) + 4 \quad \text{by Lemma 19,} \\
&\leq f^{F^\gamma(0)}(f^\gamma(f^\gamma(f^\gamma(m)))) + 4 \\
&\leq f^{F^\gamma(0)}(f^{\gamma+2}(m)) + 4 \\
&\leq f^{F^\gamma(0)}(f^{\gamma+2}(m)) + f^{\gamma+2}(0) \\
&\leq f^{F^\gamma(0)}(f^{\gamma+2}(m) + f^{\gamma+2}(0)) \\
&\leq f^{F^\gamma(0)}(f^{\gamma+3}(m)) \\
&\leq f^{F^\gamma(0)+2}(m) \\
&\leq f^{F^{\Omega \cdot \alpha + \gamma + \beta}(0)}(m). \tag{15}
\end{aligned}$$

The last inequality holds since $N(F^\gamma(0) + 2) = N(F(0)) + N(\gamma) + 2$ is bounded by $f[N(F^{\Omega \cdot \alpha + \gamma + \beta}(0))](m)$. Hence

$$\begin{aligned}
f^{F^{\Omega \cdot \alpha + \gamma + \beta_\iota}(0)+3}(m) &\leq f^{F^{\Omega \cdot \alpha + \gamma + \beta_\iota}(0)+3}(f^{F^{\Omega \cdot \alpha + \gamma + \beta}(0)}(m)) \\
&\leq f^{F^{\Omega \cdot \alpha + \gamma + \beta}(0)}(f^{F^{\Omega \cdot \alpha + \gamma + \beta}(0)}(m)) \quad \text{by (15),} \\
&\leq f^{F^{\Omega \cdot \alpha + \gamma + \beta}(0)+1}(m).
\end{aligned}$$

This together with (14) allows us to derive the sequent

$$f^{F^{\Omega \cdot \alpha + \gamma + \beta}(0)+1}[N(\iota)], F[\text{ord}(\iota)] \vdash_{\rho}^{\varphi \alpha \beta_\iota} \Gamma, A_\iota.$$

An application of (\wedge) yields $f^{F^{\Omega \cdot \alpha + \gamma + \beta}(0)+1}, F \vdash_{\rho}^{\varphi \alpha \beta} \Gamma, A$.

CASE. The last rule is (Cut): In this case there exist a formula C and an ordinal $\beta_0 < \beta$ such that $\text{rk}(C) < \rho + \omega^\alpha$, $\max\{\text{lh}(C), N(\max k_\Omega^\Pi(C)), N(\max k_\Omega^\Sigma(C))\} \leq f^\gamma(0)$, $k_\Omega(C) < F(0)$,

$$f^\gamma, F \vdash_{\rho + \omega^\alpha}^{\beta_0} \Gamma, C \quad \text{and} \quad f^\gamma, F \vdash_{\rho + \omega^\alpha}^{\beta_0} \Gamma, \neg C.$$

SIH yields $f^{F^{\Omega \cdot \alpha + \gamma + \beta_0}(0)+1}, F \vdash_{\rho}^{\varphi \alpha \beta_0} \Gamma, C$ and $f^{F^{\Omega \cdot \alpha + \gamma + \beta_0}(0)+1}, F \vdash_{\rho}^{\varphi \alpha \beta_0} \Gamma, \neg C$. If $\text{rk}(C) < \rho$, then we can apply (Cut), having the conclusion. Suppose that $\rho \leq \text{rk}(C) < \rho + \omega^\alpha$. Then there exist $l < \omega$ and $\alpha_1, \dots, \alpha_l$ such that $\alpha_l \leq \dots \leq \alpha_1 < \alpha$ and $\text{rk}(C) = \rho + \omega^{\alpha_1} + \dots + \omega^{\alpha_l}$. Let $\gamma' := F^{\Omega \cdot \alpha + \gamma + \beta_0}(0) + 2$. Then it is easy to observe that $f^{F^{\Omega \cdot \alpha + \gamma + \beta_0}(0)+1}(f^{F^{\Omega \cdot \alpha + \gamma + \beta_0}(0)+1}(m)) \leq f^{\gamma'}(m)$ for all m . This together with Cut-reduction lemma (Lemma 25) yields

$$f^{\gamma'}, F \vdash_{\rho + \omega^{\alpha_1 \cdot l}}^{\varphi \alpha \beta_0 + \varphi \alpha \beta_0} \Gamma. \tag{16}$$

Let us define ordinals ξ_n and γ_n by

$$\begin{cases} \xi_0 = \varphi \alpha \beta_0 + \varphi \alpha \beta_0, \\ \xi_{n+1} = \varphi \alpha_1 \xi_n, \end{cases} \quad \begin{cases} \gamma_0 = \gamma' = F^{\Omega \cdot \alpha + \gamma + \beta_0}(0) + 2, \\ \gamma_{n+1} = F^{\Omega \cdot \alpha_1 + \gamma_n + \xi_n}(0) + 1. \end{cases}$$

Claim. $f^{\gamma_n}, F \vdash_{\rho+\omega^{\alpha_1} \cdot (l-n)}^{\xi_n} \Gamma$. ($0 \leq n \leq l$)

We show the claim by subsidiary induction on $n \leq l$. The base case follows immediately from (16). For the induction step suppose $n < l$. Then by IH we have $f^{\gamma_n}, F \vdash_{\rho+\omega^{\alpha_1}(l-(n+1))+\omega^{\alpha_1}}^{\xi_n} \Gamma$. It is easy to see that $\{\alpha_1, \xi_n, \gamma_n\} < \Omega$ and that $\gamma < \gamma_m$ and $N(\gamma) \leq N(\gamma_m)$ for all $m \leq l$. Hence

$$\begin{cases} N(\alpha_1) \leq N(\text{rk}(C)) \leq f^\gamma(0) \leq f^{\gamma_n}(0), \\ K_\Omega \alpha_1 \subseteq K_\Omega \alpha < F(0). \end{cases}$$

Thus MIH of the lemma yields $f^{\gamma_{n+1}}, F \vdash_{\rho+\omega^{\alpha_1}(l-(n+1))}^{\xi_{n+1}} \Gamma$. \square

By the claim with $n = l$ we have $f^{\gamma_l}, F \vdash_{\rho}^{\xi_l} \Gamma$. One can show $\xi_n < \varphi\alpha\beta$ by a straightforward induction on n . Hence $\xi_l < \varphi\alpha\beta$. It remains to show that $f^{\gamma_l}(0) \leq f^{F^{\Omega \cdot \alpha + \gamma + \beta}(0)+1}(0)$. It is not difficult to check $\gamma_l < F^{\Omega \cdot \alpha + \gamma + \beta}(0) + 1$. By simultaneous induction on n we show the following (17) and (18):

$$N(\xi_n) \leq nN(\alpha_1) + 2N(\alpha) + 2N(\beta_0) + 2 + n, \quad (17)$$

$$\begin{aligned} N(\gamma_n) &\leq (n+1)N(F(0)) + \frac{1}{2}n(n+1)N(\alpha_1) \\ &\quad + (2n+1)N(\alpha) + N(\gamma) + (2n+1)N(\beta_0) + 4(n+1). \end{aligned} \quad (18)$$

For the base case

$$\begin{aligned} N(\xi_0) &\leq 2(N(\alpha) + N(\beta_0) + 1) \leq 2N(\alpha) + 2N(\beta_0) + 2, \\ N(\gamma_0) &\leq N(F(0)) + N(\alpha) + N(\gamma) + N(\beta_0) + 4. \end{aligned}$$

Let us consider the induction step. Assuming (17),

$$\begin{aligned} N(\xi_{n+1}) &= N(\alpha_1) + N(\xi_n) + 1 \\ &\leq (n+1)N(\alpha_1) + 2N(\alpha) + 2N(\beta_0) + 2 + n + 1. \end{aligned}$$

Assuming both (17) and (18),

$$\begin{aligned} N(\gamma_{n+1}) &= N(F(0)) + N(\alpha_1) + N(\gamma_n) + N(\xi_n) + 4 \\ &\leq (n+2)N(F(0)) + \left(\frac{1}{2}n(n+1) + n+1\right)N(\alpha_1) \\ &\quad + (2n+3)N(\alpha) + N(\gamma) + (2n+3)N(\beta_0) + 4(n+1) + 4 \\ &\leq (n+2)N(F(0)) + \frac{1}{2}(n+1)(n+2)N(\alpha_1) \\ &\quad + (2n+3)N(\alpha) + N(\gamma) + (2n+3)N(\beta_0) + 4(n+2). \end{aligned}$$

Let us observe that

$$\begin{aligned} N(\text{rk}(C)) &\leq N(\max \mathbf{k}(C)) + \text{lh}(C) \quad \text{by Lemma 15.3,} \\ &\leq f^\gamma(0) + f^\gamma(0) \quad \text{since } \text{lh}(C) \leq f^\gamma(0), \\ &\leq f^\gamma(f^\gamma(0)). \end{aligned} \quad (19)$$

Hence $l \leq \text{rk}(C) \leq f^\gamma(f^\gamma(0)) \leq f^{F^\gamma(0)}(0)$ since $\gamma < F^\gamma(0)$ and $N(\gamma) \leq N(F^\gamma(0))$. Further $\max\{F(0), N(\alpha), N(\beta_0)\} \leq f^\gamma(0) \leq f^{F^\gamma(0)}$ by assumption and $N(\gamma) \leq f^{F^\gamma(0)}(0)$ by Lemma 19. From these and (18),

$$N(\gamma_l) \leq (f^{F^\gamma(0)}(0))^3 + 6(f^{F^\gamma(0)}(0))^2 + 8 \cdot f^{F^\gamma(0)}(0) + 4. \quad (20)$$

On the other hand, from Example 18, one can see that $m^3 + 6m^2 + 8m + 4 \leq f^{F^\gamma(0)}(m)$ holds. Hence by (20),

$$N(\gamma_l) \leq f^{F^\gamma(0)}(f^{F^\gamma(0)}(0)) \leq f^{F^{\Omega \cdot \alpha + \gamma + \beta}(0)}(0) \leq f(f^{F^{\Omega \cdot \alpha + \gamma + \beta}(0)}(0)). \quad (21)$$

Hence

$$\begin{aligned} f^{\gamma_l}(0) &\leq f^{\gamma_l}(f^{F^{\Omega \cdot \alpha + \gamma + \beta}(0)}(0)) \\ &\leq f^{F^{\Omega \cdot \alpha + \gamma + \beta}(0)}(f^{F^{\Omega \cdot \alpha + \gamma + \beta}(0)}(0)) \quad \text{by (21),} \\ &\leq f^{F^{\Omega \cdot \alpha + \gamma + \beta}(0)+1}(0). \end{aligned}$$

This allows us to conclude $f^{F^{\Omega \cdot \alpha + \gamma + \beta}(0)+1}, F \vdash_{\rho}^{\varphi \alpha \beta} \Gamma$. \square

Definition 28. For each \mathcal{L}^* -formula B let B^α be the result of replacing in B every occurrence of $P_{\mathcal{A}}^{<\Omega}$ by $P_{\mathcal{A}}^{<\alpha}$.

Lemma 29 (Boundedness). Assume that $f, F \vdash_{\rho}^{\alpha} \Gamma, A$. Then for all ξ if $\alpha \leq \xi \leq F(0)$, $N(\xi) \leq f(0)$ and $K_{\Omega}\xi < F(0)$, then $f, F \vdash_{\rho}^{\alpha} \Gamma, A^\xi$.

Proof. The claim is trivial if $F(0) < \alpha$. Assume that $\alpha \leq F(0)$ and $f, F \vdash_{\rho}^{\alpha} \Gamma, A$. By induction on α we show that for all ξ if $\alpha \leq \xi \leq F(0)$, then $f, F \vdash_{\rho}^{\alpha} \Gamma, A^\xi$.

CASE. The last rule is (\bigvee) : If A is not the principal formula of last rule (\bigvee) , then the claim follows immediately from IH. Suppose that $A \simeq \bigvee_{i \in J} A_i$ is the principal formula of (\bigvee) . Then there exist $\alpha_0 < \alpha$ and $i_0 \in J$ such that $\text{ord}(\iota_0) < \alpha$ and $f, F \vdash_{\rho}^{\alpha_0} \Gamma, A, A_{i_0}$. Let $\alpha \leq \xi \leq F(0)$. Then IH yields $f, F \vdash_{\rho}^{\alpha_0} \Gamma, A^\xi, A_{i_0}$. If $A \not\equiv P_{\mathcal{A}}^{<\Omega} t$, then another application of IH and an application of (\bigvee) yield $f, F \vdash_{\rho}^{\alpha} \Gamma, A^\xi$. Consider the case that $A \equiv P_{\mathcal{A}}^{<\Omega} t \simeq \bigvee_{\mu < \Omega} \mathcal{A}(P_{\mathcal{A}}^{<\mu}, t)$. In this subcase $A_{\mu_0} \simeq \mathcal{A}(P_{\mathcal{A}}^{<\mu_0}, t)$. Since $\mu_0 = \text{ord}(\mu_0) < \alpha \leq \xi$, we can apply (\bigvee) and then obtain $f, F \vdash_{\rho}^{\alpha} \Gamma, P_{\mathcal{A}}^{<\xi}$.

CASE. The last rule is (\bigwedge) : In this case for all $i \in J$ there exists $\alpha_i < \alpha$ such that $f[N(\iota)], F[\text{ord}(\iota)] \vdash_{\rho}^{\alpha_i} \Gamma'$ for a certain Γ' . Let us observe that $F(0) \leq F[\text{ord}(\iota)](0)$. Hence, if A is not the principal formula of (\bigwedge) , then the claim follows immediately from IH. Suppose that A is the principal formula of (\bigwedge) . Then $A \simeq \bigwedge_{i \in J} A_i \in \Gamma$ and $\Gamma' \equiv \Gamma, A, A_i$. Let $\alpha \leq \xi \leq F(0)$. Then IH yields $f[N(\iota)], F[\text{ord}(\iota)] \vdash_{\rho}^{\alpha_i} \Gamma, A^\xi, A_i$. If $A \not\equiv \neg P_{\mathcal{A}}^{<\Omega} t$, then another application of IH and an application of (\bigwedge) yield $f, F \vdash_{\rho}^{\alpha} \Gamma, A^\xi$. If $A \equiv \neg P_{\mathcal{A}}^{<\Omega} t \simeq \bigwedge_{\mu < \Omega} \neg \mathcal{A}(P_{\mathcal{A}}^{<\mu}, t)$, then an application of (\bigwedge) with $\mu \leq \xi \leq F(0) < \Omega$ yields $f, F \vdash_{\rho}^{\alpha} \Gamma, \neg P_{\mathcal{A}}^{<\xi}$.

CASE. The last rule is (Cl_{Ω}) : If A is not the principal formula, then the claim again follows from IH. Let us consider the case that A is the principal

formula of the last rule (Cl_Ω) with a premise $f, F \vdash_{\rho}^{\alpha_0} \Gamma, P_{\mathcal{A}}^{<\Omega} t, \mathcal{A}(P_{\mathcal{A}}^{<\Omega}, t)$ for some $\alpha_0 < \alpha$ where $A \equiv P_{\mathcal{A}}^{<\Omega} t$. Let $\alpha \leq \xi \leq F(0)$. An application of IH yields $f, F \vdash_{\rho}^{\alpha_0} \Gamma, P_{\mathcal{A}}^{<\xi}, \mathcal{A}(P_{\mathcal{A}}^{<\Omega}, t)$. Another application of IH yields $f, F \vdash_{\rho}^{\alpha_0} \Gamma, P_{\mathcal{A}}^{<\xi}, \mathcal{A}(P_{\mathcal{A}}^{<\alpha_0}, t)$. Let us observe that $\text{ord}(\alpha_0) = \alpha_0 < \alpha$, $N(\alpha_0) \leq f(0)$, and $\text{ord}(\alpha_0) = \alpha_0 < \alpha \leq F(0)$. Hence we can apply (\vee) with $\alpha_0 < \alpha \leq \xi$, concluding $f, F \vdash_{\rho}^{\alpha} \Gamma, P_{\mathcal{A}}^{<\xi}$. \square

We will write $f, F \vdash_{\cdot}^{\alpha} \Gamma$ instead of $f, F \vdash_{\alpha}^{\alpha} \Gamma$.

Lemma 30 (Impredicative Cut-elimination).

If $f, F \vdash_{\Omega+1}^{\alpha} \Gamma$, then $f^{F^{\alpha}(0)+1}, F^{\alpha+1} \vdash_{\cdot}^{F^{\alpha}(0)} \Gamma$.

Proof. By induction on α . It is easy to check that $f(0) \leq f^{F^{\alpha}(0)+1}(0)$ and $F(0) \leq F^{\alpha}(0)$. It also holds that $K_{\Omega} F^{\alpha}(0) = \{F^{\alpha}(0)\} < F^{\alpha+1}(0)$. Further,

$$\begin{aligned} N(F^{\alpha+1}(0)) &= N(F(0)) + N(\alpha) + 1 \leq f(0) + f(0) + 1 \\ &\leq f(f(0)) + 1 \\ &\leq f^{F^{\alpha}(0)+1}(0). \end{aligned}$$

And hence $N(F^{\alpha}(0)) < N(F^{\alpha+1}(0)) \leq f^{F^{\alpha}(0)+1}(0)$ in particular. Let (\mathcal{J}) denote the last rule that forms $f, F \vdash_{\Omega+1}^{\alpha} \Gamma$.

CASE. (\mathcal{J}) is (Cut) with a cut formula C : In this case (\mathcal{J}) has two premises $f, F \vdash_{\Omega+1}^{\alpha_0} \Gamma, C$ and $f, F \vdash_{\Omega+1}^{\alpha_0} \Gamma, \neg C$ for some $\alpha_0 < \alpha$. IH yields that

$$f^{F^{\alpha_0}(0)+1}, F^{\alpha_0+1} \vdash_{\cdot}^{F^{\alpha_0}(0)} \Gamma, C, \quad (22)$$

$$f^{F^{\alpha_0}(0)+1}, F^{\alpha_0+1} \vdash_{\cdot}^{F^{\alpha_0}(0)} \Gamma, \neg C. \quad (23)$$

Let us observe that $F^{\alpha_0}(0) < F^{\alpha}(0)$ since $K_{\Omega} \alpha_0 < F(0) \leq F^{\alpha}(0)$. Similarly $F^{\alpha_0+1}(0) < F^{\alpha+1}(0)$ holds. Further

$$\begin{aligned} N(F^{\alpha_0}(0) + 1) &= N(F(0)) + N(\alpha_0) + 1 \\ &\leq N(F^{\alpha}(0)) + f(0) + 1 \quad \text{since } N(\alpha_0) \leq f(0), \\ &\leq f(N(F^{\alpha}(0) + 1)) = f[N(F^{\alpha}(0) + 1)](0). \end{aligned}$$

Hence $f^{F^{\alpha_0}(0)+1}(0) < f^{F^{\alpha}(0)+1}(0)$.

SUBCASE. $\text{rk}(C) < \Omega$. By Lemma 15.3 $\text{rk}(C) = \text{rk}(\neg C) \leq \omega \cdot (\max k_{\Omega}^H(\neg C)) + \text{lh}(\neg C) < F(0)$ since $k_{\Omega}^H(\neg C) \subseteq k_{\Omega}(C) < F(0)$. Hence $\text{rk}(C) < F(0) \leq F^{\alpha}(0)$. This together with the two sequents (22) and (23) allows us to deduce other two sequents $f^{F^{\alpha}(0)+1}, F^{\alpha+1} \vdash_{\cdot}^{F^{\alpha}(0)} \Gamma, C$ and $f^{F^{\alpha}(0)+1}, F^{\alpha+1} \vdash_{\cdot}^{F^{\alpha}(0)} \Gamma, \neg C$. We can apply (Cut) to these two sequents, concluding $f^{F^{\alpha}(0)+1}, F^{\alpha+1} \vdash_{\cdot}^{F^{\alpha}(0)} \Gamma$.

SUBCASE. $\text{rk}(C) = \Omega$. In this case $C \equiv P_{\mathcal{A}}^{<\Omega} t$ by Lemma 15.4. Let us observe the following.

1. $N(F^{\alpha_0}(0)) = N(F(0)) + N(\alpha_0) \leq f(0) + f(0) \leq f(f(0)) \leq f^{F^{\alpha_0}(0)}(0)$.
2. $K_{\Omega} F^{\alpha_0}(0) = \{F^{\alpha_0}(0)\} < F^{\alpha_0+1}(0)$.

Applying Boundedness lemma (Lemma 29) to the sequent (22) yields the sequent $f^{F^{\alpha_0}(0)}, F^{\alpha_0+1} \vdash_{\cdot}^{F^{\alpha_0}(0)} \Gamma, P_{\mathcal{A}}^{<F^{\alpha_0}(0)}$. As in the previous subcase this induces the sequent

$$f^{F^{\alpha_0}(0)+1}, F^{\alpha_0+1} \vdash_{\cdot}^{F^{\alpha_0}(0)} \Gamma, P_{\mathcal{A}}^{<F^{\alpha_0}(0)}. \quad (24)$$

On the other hand applying Inversion lemma (Lemma 24) to the sequent (23) yields the sequent

$$f^{F^{\alpha_0}(0)+1}[N(F^{\alpha_0}(0))], F^{\alpha_0+1}[F^{\alpha_0}(0)] \vdash_{\cdot}^{F^{\alpha_0}(0)} \Gamma, \neg P_{\mathcal{A}}^{<F^{\alpha_0}(0)}.$$

By Property 1 we can see that $f^{F^{\alpha_0}(0)+1}[N(F^{\alpha_0}(0))](0) \leq f^{F^{\alpha_0}(0)+1}(f^{F^{\alpha_0}(0)}(0)) \leq f^{F^{\alpha_0}(0)+1}(0)$ and $F^{\alpha_0+1}[F^{\alpha_0}(0)](0) \leq F^{\alpha_0+1}(0)$. These observations induce the sequent

$$f^{F^{\alpha_0}(0)+1}, F^{\alpha_0+1} \vdash_{\cdot}^{F^{\alpha_0}(0)} \Gamma, \neg P_{\mathcal{A}}^{<F^{\alpha_0}(0)}. \quad (25)$$

By definition $\text{rk}(P_{\mathcal{A}}^{<F^{\alpha_0}(0)}) = \text{rk}(\neg P_{\mathcal{A}}^{<F^{\alpha_0}(0)}) = F^{\alpha_0}(0) < F^{\alpha_0}(0)$. Now by an application of (Cut) to the two sequents (24) and (25) we can derive the desired sequent $f^{F^{\alpha_0}(0)+1}, F^{\alpha_0+1} \vdash_{\cdot}^{F^{\alpha_0}(0)} \Gamma$.

CASE. (\mathcal{J}) is (\bigwedge) with a principal formula $A \simeq \bigwedge_{\iota \in J} A_{\iota} \in \Gamma$: In this case $\forall \iota \in J, \exists \alpha_{\iota} < \alpha$ s.t. $f[N(\iota)], F[\text{ord}(\iota)] \vdash_{\Omega+1}^{\alpha_{\iota}} \Gamma, A_{\iota}$. IH yields the sequent

$$f[N(\iota)]^{F[\text{ord}(\iota)]^{\alpha_{\iota}}(0)+1}, F[\text{ord}(\iota)]^{\alpha_{\iota}+1} \vdash_{\cdot}^{F[\text{ord}(\iota)]^{\alpha_{\iota}}(0)} \Gamma, A_{\iota}$$

for all $\iota \in J$. In the same way as we showed the claim in the proof of Lemma 26 (p. 14), one can show that for all $\iota \in J$

$$\begin{aligned} f[N(\iota)]^{F[\text{ord}(\iota)]^{\alpha_{\iota}}(0)+1}(0) &\leq f^{F^{\alpha_0}(0)+1}[N(\iota)](0), \\ F[\text{ord}(\iota)]^{\alpha_{\iota}+1}(0) &\leq F^{\alpha_0+1}[\text{ord}(\iota)](0). \end{aligned}$$

These enable us to deduce the sequent

$$f^{F^{\alpha_0}(0)+1}[N(\iota)], F^{\alpha_0+1}[\text{ord}(\iota)] \vdash_{F^{\alpha_0}(0)}^{F[\text{ord}(\iota)]^{\alpha_{\iota}}(0)} \Gamma, A_{\iota}$$

for all $\iota \in J$. Since $F[\text{ord}(\iota)]^{\alpha_{\iota}}(0) < F^{\alpha_0}(0)$ for all $\iota \in J$, we can apply (\bigwedge) to this sequent, concluding $f^{F^{\alpha_0}(0)+1}, F^{\alpha_0+1} \vdash_{\cdot}^{F^{\alpha_0}(0)} \Gamma$. \square

Lemma 31 (Witnessing). *For each $j < l$ let $B_j(x)$ be a Δ_0^0 - \mathcal{L}_{PA} -formula such that $\text{FV}(B_j(x)) = \{x\}$. Let $\Gamma \equiv \exists x_0 B_0(x_0), \dots, \exists x_{l-1} B_{l-1}(x_{l-1})$. If $f, F \vdash_0^{\alpha} \Gamma$ for some $\alpha \in \mathcal{OT}(\mathcal{F})$, then there exists a sequence $\langle m_0, \dots, m_{l-1} \rangle$ of naturals such that $\max\{m_j \mid j < l\} \leq f(0)$ and $B_0(\underline{m_0}) \vee \dots \vee B_{l-1}(\underline{m_{l-1}})$ is true in the standard model \mathbb{N} of PA.*

Proof. By induction on α . The derivation forming $f, F \vdash_0^{\alpha} \Gamma$ contains no (Cut) rules. Hence the last inference rule should be (\bigvee) . Thus there exist an ordinal $\alpha_0 < \alpha$ and a (closed) term $t \in \mathcal{T}(\mathcal{L}_{\text{ID}_1})$ such that $N(t) \leq f(0)$ and $f, F \vdash_0^{\alpha_0} \Gamma, B_{l-1}(t)$. By IH there exists a sequence $\langle m_0, \dots, m_{l-1} \rangle$ of naturals such that $\max\{m_j \mid j < l\} \leq f(0)$ and $B_0(\underline{m_0}) \vee \dots \vee B_{l-1}(\underline{m_{l-1}}) \vee B_{l-1}(t)$ is true in

\mathbb{N} . If $B_0(\underline{m_0}) \vee \dots \vee B_{l-1}(\underline{m_{l-1}})$ is already true in \mathbb{N} , then $\langle m_0, \dots, m_{l-1} \rangle$ is the desired sequence. Suppose that $B_0(\underline{m_0}) \vee \dots \vee B_{l-1}(\underline{m_{l-1}})$ is not true in \mathbb{N} . Then $B_{l-1}(t)$ must be true. Hence $B_{l-1}(\underline{\text{val}(t)})$ is also true. By definition, $\text{val}(t) = N(t) \leq f(0)$, and hence $\langle m_0, \dots, m_{l-2}, \text{val}(t) \rangle$ is the desired sequence. \square

5 Embedding ID_1 into ID_1^∞

In this section we embed the theory ID_1 into the infinitary system ID_1^∞ . Following conventions in the previous section we use the symbol f to denote a strict increasing function $f : \mathbb{N} \rightarrow \mathbb{N}$ that enjoys the conditions (f.1) and (f.2) (p. 8). Let us recall that the function symbol $\mathbf{E} \in \mathcal{F}$ denotes the function $\mathbf{E} : \Omega \rightarrow \Omega$ such that $\mathbf{E}(\alpha) = \min\{\xi < \Omega \mid \alpha < \xi \text{ and } \xi = \omega^\xi\}$. It is easy to see that the condition (HYP(\mathbf{E})) holds since $\mathbf{E}(\xi) = \varepsilon_0 \leq \mathbf{E}(0)$ for all $\xi < \mathbf{E}(0) = \varepsilon_0$.

Lemma 32 (Tautology lemma). *Let $s, t \in \mathcal{T}(\mathcal{L}_{\text{ID}_1})$, Γ be a sequent of \mathcal{L}^* -sentences, and $A(x)$ be an \mathcal{L}^* -formula such that $\text{FV}(A) = \{x\}$. If $\text{val}(s) = \text{val}(t)$, then*

$$f[n], \mathbf{E}[\mathbf{k}_\Omega(A)] \vdash_0^{\text{rk}(A) \cdot 2} \Gamma, \neg A(s), A(t), \quad (26)$$

where $n := \max\{N(\text{rk}(A)), N(\max \mathbf{k}_\Omega^H(A)), N(\max \mathbf{k}_\Omega^\Sigma(A))\}$.

Proof. By induction on $\text{rk}(A)$. Let n denote the maximal among $N(\text{rk}(A))$, $N(\max \mathbf{k}_\Omega^H(A))$ and $N(\max \mathbf{k}_\Omega^\Sigma(A))$. From Lemma 15.3 one can check that the condition $\text{HYP}(f[n]; \mathbf{E}[\mathbf{k}_\Omega(A)]; \text{rk}(A) \cdot 2)$ holds. If $\text{rk}(A) = 0$, then A is an $\mathcal{L}_{\text{ID}_1}$ -literal, and hence (26) is an instance of (Ax1). Suppose that $\text{rk}(A) > 0$. Without loss of generality we can assume that $A \simeq \bigvee_{\iota \in J} A_\iota$. Let $\iota \in J$. By Lemma 15.5 let us observe that $N(\text{rk}(A_\iota) \cdot 2) < 2\{N(\text{rk}(A)), N(\iota)\} \leq f[N(\text{rk}(A))][N(\iota)](0) \leq f[n](0)$ since $2m+1 \leq f(m)$ for all m by the condition (f.1). Further by Lemma 15.1 $K_\Omega(\text{rk}(A_\iota) \cdot 2) \subseteq \mathbf{k}_\Omega(A) \cup \{\text{ord}(\iota)\} \leq \mathbf{E}[\mathbf{k}_\Omega(A)][\text{ord}(\iota)]$. Summing up, we have the condition

$$\text{HYP}(f[n][N(\iota)]; \mathbf{E}[\mathbf{k}_\Omega(A)][\text{ord}(\iota)]; \text{rk}(A_\iota) \cdot 2).$$

Hence by IH we can obtain the sequent

$$f[n][N(\iota)], \mathbf{E}[\mathbf{k}_\Omega(A)][\text{ord}(\iota)] \vdash_0^{\text{rk}(A_\iota) \cdot 2} \Gamma, \neg A_\iota(s), A_\iota(t). \quad (27)$$

It is not difficult to see $\text{ord}(\iota) \leq \text{rk}(A_\iota) < \text{rk}(A_\iota) \cdot 2 + 1$ and $N(\text{rk}(A_\iota) \cdot 2 + 1) = N(\text{rk}(A_\iota) \cdot 2) + 1 \leq f[N(\text{rk}(A))][N(\iota)](0) \leq f[n](0)$. This allows us to apply (\vee) to the sequent (27) yielding

$$f[n][N(\iota)], \mathbf{E}[\mathbf{k}_\Omega(A)][\text{ord}(\iota)] \vdash_0^{\text{rk}(A_\iota) \cdot 2 + 1} \Gamma, \neg A_\iota(s), A(t).$$

We can see that $\text{rk}(A_\iota) \cdot 2 + 1 < \text{rk}(A) \cdot 2$, $N(\max \mathbf{k}_\Omega^H(A)) \leq f[n](0)$ and $\mathbf{k}_\Omega^H(A) < \mathbf{E}[\mathbf{k}_\Omega(A)]$. Hence we can apply (\wedge) concluding (26). \square

Lemma 33. *Let B_j be an $\mathcal{L}_{\text{ID}_1}$ -sentence for each $j = 0, \dots, l$. Suppose that $(\neg B_0) \vee \dots \vee (\neg B_{l-1}) \vee B_l$ is a logical consequence in the first order predicate logic with equality. Then there exists a natural $k < \omega$ such that $f[m+k], E \vdash_0^{\Omega \cdot 2 + k} \{B_j \mid 0 \leq j \leq l\}$, where $m = \max\{N(\text{rk}(B_j)) \mid j = 0, 1, \dots, l\}$.*

Proof. Let B_j be an $\mathcal{L}_{\text{ID}_1}$ -sentence for each $j = 0, \dots, l-1$ and suppose that $B_0 \vee \dots \vee B_{l-1}$ is a logical consequence in the first order predicate logic with equality. Then we can find a cut-free proof of the sequent $\{B_j \mid 0 \leq j \leq l-1\}$ in an **LK**-style sequent calculus. More precisely we can find a cut-free proof P of $\{B_j \mid 0 \leq j \leq l-1\}$ in the sequent calculus **G3_m**. (See the book [17] of Troelstra and Schwichtenberg for the definition.) Let h denote the tree height of the cut-free proof P . Then by induction on h one can find a witnessing natural k such that $f[m+k], F \vdash_0^\alpha \{B_j \mid 0 \leq j \leq l-1\}$ for all $\alpha \geq \Omega + k$. In case $h = 0$ Tautology lemma (Lemma 32) can be applied since for any $\mathcal{L}_{\text{ID}_1}$ -sentence A , $\text{rk}(A) \in \omega \cup \{\Omega + k \mid k < \omega\}$ and $k^\Pi(A) \cup k^\Sigma(A) = k(A) \subseteq \{0, \Omega\}$, and hence $k_\Omega(A) = \{0\}$ and $\max\{N(\max k_\Omega^\Pi(A)), N(\max k_\Omega^\Sigma(A))\} = 0$. \square

Lemma 34. *Let $m \in \mathbb{N}$ and $A(x)$ be an $\mathcal{L}_{\text{ID}_1}$ -formula such that $\text{FV}(A(x)) = \{x\}$. Then for any $t \in \mathcal{T}(\mathcal{L}_{\text{ID}_1})$ and for any sequent Γ of $\mathcal{L}_{\text{ID}_1}$ -sentences, if $\text{val}(t) = m$, then*

$$f[N(\text{rk}(A)) + m], E \vdash_0^{(\text{rk}(A)+m) \cdot 2} \Gamma, \neg A(0), \neg \forall x(A(x) \rightarrow A(S(x))), A(t). \quad (28)$$

Proof. By induction on m . The base case $\text{val}(t) = m = 0$ follows from Tautology lemma (Lemma 32). For the induction step suppose $\text{val}(t) = m + 1$. Fix a sequent Γ of $\mathcal{L}_{\text{ID}_1}$ -sentences. Then (28) holds by IH. On the other hand again by Tautology lemma,

$$f[N(\text{rk}(A))], E \vdash_0^{\text{rk}(A) \cdot 2} \Gamma, \neg A(0), \exists x(A(x) \wedge \neg A(S(x))), A(\underline{m}), \neg A(\underline{m}). \quad (29)$$

An application of (\wedge) to the two sequents (28) and (29) yields

$$f[N(\alpha_m)], E \vdash_0^{\alpha_m \cdot 2 + 1} \Gamma, \neg A(0), \exists x(A(x) \wedge \neg A(S(x))), A(t), A(\underline{m}) \wedge \neg A(\underline{m}),$$

where $\alpha_m := \text{rk}(A) + m$. The final application of (\vee) yields

$$f[N(\text{rk}(A)) + m + 1], F \vdash_0^{(\text{rk}(A)+m+1) \cdot 2} \Gamma, \neg A(0), \exists x(A(x) \wedge \neg A(S(x))), A(t).$$

\square

Lemma 35. *Let $\xi \leq \Omega$, $F(x)$ be an $\mathcal{L}_{\text{ID}_1}$ -formula such that $\text{FV}(F(x)) = \{x\}$ and $B(X)$ be an X -positive $\mathcal{L}_{\text{PA}}(X)$ -formula such that $\text{FV}(B) = \emptyset$. Then*

$$f[N(\sigma + \alpha + 1)], E[K_\Omega \xi] \vdash_0^{(\sigma + \alpha + 1) \cdot 2} \Gamma, \neg \forall x(\mathcal{A}(F, x) \rightarrow F(x)), \neg B(P_{\mathcal{A}}^{<\xi}), B(F),$$

where $\sigma := \text{rk}(F)$ and $\alpha := \text{rk}(B(P_{\mathcal{A}}^{<\xi}))$.

Proof. By main induction on ξ and side induction on $\text{rk}(B(P_{\mathcal{A}}^{<\xi}))$. Let $\text{Cl}_{\mathcal{A}}(F) := \neg\forall x(\mathcal{A}(F, x) \rightarrow F(x)) \equiv \exists x(\mathcal{A}(F, x) \wedge \neg F(x))$. The argument splits into several cases depending on the shape of the formula $B(X)$.

CASE. $B(X)$ is an \mathcal{L}_{PA} -literal: In this case B does not contain the set free variable X , and hence Tautology lemma (Lemma 32) can be applied.

CASE. $B \equiv X(t)$ for some $t \in \mathcal{T}(\mathcal{L}_{\text{ID}_1})$: In this case $\neg B(P_{\mathcal{A}}^{<\xi}) \equiv \neg P_{\mathcal{A}}^{<\xi} t \equiv \bigwedge_{\eta < \xi} \neg \mathcal{A}(P_{\mathcal{A}}^{<\eta}, t)$. Let $\eta < \xi$. Then by MIH

$$f[N(\sigma + \alpha_{\eta} + 1)], E[K_{\Omega}\eta] \vdash_0^{(\sigma + \alpha_{\eta} + 1) \cdot 2} \Gamma, \text{Cl}_{\mathcal{A}}(F), \neg \mathcal{A}(P_{\mathcal{A}}^{<\eta}, t), \mathcal{A}(F, t), F(t)$$

where $\alpha_{\eta} := \text{rk}(\mathcal{A}(P_{\mathcal{A}}^{<\eta}, t))$. We note that $\eta < \xi \leq \Omega$ and hence $K_{\Omega}\eta = \{\eta\} = \{\text{ord}(\eta)\}$. Hence this yields the sequent

$$f[N(\sigma + \alpha)][N(\eta)], E[\text{ord}(\eta)] \vdash_0^{(\sigma + \alpha_{\eta} + 1) \cdot 2} \Gamma, \text{Cl}_{\mathcal{A}}(F), \neg \mathcal{A}(P_{\mathcal{A}}^{<\eta}, t), \mathcal{A}(F, t), F(t).$$

An application of (\bigwedge) yields the sequent

$$f[N(\sigma + \alpha)], E[K_{\Omega}\xi] \vdash_0^{(\sigma + \alpha) \cdot 2} \Gamma, \text{Cl}_{\mathcal{A}}(F), \neg P_{\mathcal{A}}^{<\xi} t, \mathcal{A}(F, t), F(t). \quad (30)$$

On the other hand by Tautology lemma (Lemma 32),

$$f[N(\sigma + \alpha)], E[K_{\Omega}\xi] \vdash_0^{\text{rk}(F) \cdot 2} \Gamma, \text{Cl}_{\mathcal{A}}(F), \neg P_{\mathcal{A}}^{<\xi} t, \neg F(t), F(t). \quad (31)$$

Another application of (\bigwedge) to the two sequents (30) and (31) yields the sequent

$$f[N(\sigma + \alpha + 1)], E[K_{\Omega}\xi] \vdash_0^{(\sigma + \alpha) \cdot 2 + 1} \Gamma, \text{Cl}_{\mathcal{A}}(F), \neg P_{\mathcal{A}}^{<\xi} t, \mathcal{A}(F, t) \wedge \neg F(t), F(t).$$

An application of (\bigvee) allows us to conclude

$$f[N(\sigma + \alpha + 1)], E[K_{\Omega}\xi] \vdash_0^{(\sigma + \alpha + 1) \cdot 2} \Gamma, \text{Cl}_{\mathcal{A}}(F), \neg P_{\mathcal{A}}^{<\xi} t, F(t).$$

CASE. $B(X) \equiv \forall y B_0(X, y)$ for some \mathcal{L}_{PA} -formula $B_0(X, y)$: Let α_0 denote the ordinal $\text{rk}(B_0(P_{\mathcal{A}}^{<\xi}, \underline{0}))$. Then $\alpha = \alpha_0 + 1$. By the definition of the rank function rk , $\alpha_0 = \text{rk}(B_0(P_{\mathcal{A}}^{<\xi}, t))$ for all $t \in \mathcal{T}(\mathcal{L}_{\text{ID}_1})$. Fix a closed term $t \in \mathcal{T}(\mathcal{L}_{\text{ID}_1})$. Then from SIH we have the sequent

$$f[N(\sigma + \alpha + 1)], E[K_{\Omega}\xi] \vdash_0^{(\sigma + \alpha) \cdot 2} \Gamma, \text{Cl}_{\mathcal{A}}(F), \neg B_0(P_{\mathcal{A}}^{<\xi}, t), B_0(P_{\mathcal{A}}^{<\xi}, t).$$

An application of (\bigvee) yields the sequent

$$f[N(\sigma + \alpha + 1)], E[K_{\Omega}\xi] \vdash_0^{(\sigma + \alpha) \cdot 2 + 1} \Gamma, \text{Cl}_{\mathcal{A}}(F), \neg \forall y B_0(P_{\mathcal{A}}^{<\xi}, y), B_0(P_{\mathcal{A}}^{<\xi}, t).$$

And an application of (\bigwedge) allows us to conclude.

The other cases can be treated in similar ways. \square

Lemma 36. 1. $f[N(\text{rk}(\mathcal{A}(P_{\mathcal{A}}^{<\Omega}, \underline{0})) + 1], E \vdash_0^{\Omega \cdot 2 + \omega} \forall x(\mathcal{A}(P_{\mathcal{A}}^{<\Omega}, x) \rightarrow P_{\mathcal{A}}^{<\Omega} x)$.
 2. $f[3+l], E \vdash_0^{\Omega \cdot 2 + \omega} \forall \mathbf{y}[\forall x\{\mathcal{A}(F(\cdot, \mathbf{y}), x) \rightarrow F(x, \mathbf{y})\} \rightarrow \forall x\{P_{\mathcal{A}}^{<\Omega} x \rightarrow F(x, \mathbf{y})\}]$,
 where $\mathbf{y} = y_0, \dots, y_{l-1}$.

Proof. PROPERTY 1. Let $\alpha = \text{rk}(\mathcal{A}(P_{\mathcal{A}}^{<\Omega}, 0))$ and $t \in \mathcal{T}(\mathcal{L}_{\text{ID}_1})$. By the definition of rk we can find a natural $k < \omega$ such that $\alpha = \text{rk}(\mathcal{A}(P_{\mathcal{A}}^{<\Omega}, t)) = \Omega + k$. This implies $\text{k}(\mathcal{A}(P_{\mathcal{A}}^{<\Omega}, t)) = \{0, \Omega\}$ and hence $\text{k}_{\Omega}(\mathcal{A}(P_{\mathcal{A}}^{<\Omega}, t)) = \{0\} < \text{E}(0)$. By Tautology lemma (Lemma 32),

$$f[N(\alpha)], \text{E} \vdash_0^{\alpha \cdot 2} P_{\mathcal{A}}^{<\Omega} t, \neg \mathcal{A}(P_{\mathcal{A}}^{<\Omega}, t), \mathcal{A}(P_{\mathcal{A}}^{<\Omega}, t).$$

Since $\Omega < \Omega \cdot 2 + k + 1 = \alpha \cdot 2 + 1$, we can apply the closure rule (Cl_{Ω}) obtaining the sequent

$$f[N(\alpha)], \text{E} \vdash_0^{\Omega \cdot 2 + k + 1} \neg \mathcal{A}(P_{\mathcal{A}}^{<\Omega}, t), P_{\mathcal{A}}^{<\Omega} t.$$

An application of (\wedge) followed by an application of (\vee) enables us to conclude

$$f[N(\alpha) + 1], \text{E} \vdash_0^{\Omega \cdot 2 + \omega} \forall x (\mathcal{A}(P_{\mathcal{A}}^{<\Omega}, x) \rightarrow P_{\mathcal{A}}^{<\Omega} x).$$

PROPERTY 2. By definition $\text{rk}(P_{\mathcal{A}}^{<\Omega}) = \omega \cdot \Omega = \Omega$. On the other hand $\text{rk}(F) < \omega$ and hence $(\text{rk}(F) + \text{rk}(P_{\mathcal{A}}^{<\Omega}) + 1) \cdot 2 = \Omega \cdot 2 + 2$. Let $s, \mathbf{t} = s, t_0, \dots, t_{l-1} \in \mathcal{T}(\mathcal{L}_{\text{ID}_1})$. Then by the previous lemma (Lemma 35)

$$f[2], \text{E} \vdash_0^{\Omega \cdot 2 + 1} \neg \forall x (\mathcal{A}(F(\cdot, \mathbf{t}), x) \rightarrow F(x, \mathbf{t})), \neg P_{\mathcal{A}}^{<\Omega} t, F(s, \mathbf{t})$$

since $N(\Omega + 1) = 2$. It is not difficult to see that applications of (\vee) , (\wedge) and (\vee) in this order yield the sequent

$$f[3], \text{E} \vdash_0^{\Omega \cdot 2 + 5} \forall x (\mathcal{A}(F(\cdot, \mathbf{t}), x) \rightarrow F(x, \mathbf{t})) \rightarrow \forall x (P_{\mathcal{A}}^{<\Omega} x \rightarrow F(x, \mathbf{t}))$$

Finally, l -fold application of (\wedge) allows us to conclude. \square

Let us recall that \mathbf{s} denotes the numerical successor $m \mapsto m + 1$.

Theorem 37. Let $A \equiv \forall \mathbf{x} \exists y B(\mathbf{x}, y)$ be a Π_2^0 -sentence for a Δ_0^0 -formula $B(\mathbf{x}, y)$ such that $\text{FV}(B(\mathbf{x}, y)) = \{\mathbf{x}, y\}$. If $\text{ID}_1 \vdash A$, then we can an ordinal term $\alpha \in \mathcal{OT}(\mathcal{F}) \upharpoonright \Omega$ built up without the Veblen function symbol φ such that for all $\mathbf{m} = m_0, \dots, m_{l-1} \in \mathbb{N}$ there exists $n \leq \mathbf{s}^{\alpha}(m_0 + \dots + m_{l-1})$ such that $B(\mathbf{m}, n)$ is true in the standard model \mathbb{N} of PA.

Proof. Assume $\text{ID}_1 \vdash A$. Then there exist ID_1 -axioms A_1, \dots, A_k such that $(\neg A_1) \vee \dots \vee (\neg A_k) \vee A$ is a logical consequence in the first order predicate logic with equality. Hence by Lemma 33,

$$f[c_0], \text{E} \vdash_0^{\Omega \cdot 3} \neg A_1, \dots, \neg A_k, A$$

for some constant $c_0 < \omega$ depending on $N(\text{rk}(A_1)), \dots, N(\text{rk}(A_k)), N(\text{rk}(A))$ and depending also on the tree height of a cut-free **LK**-derivation of the sequent $\neg A_1, \dots, \neg A_k, A$. By Lemma 34 and 36, for each $j = 1, \dots, k$, there exists a constant c_j depending on $\text{rk}(A_j)$ such that $f[c_j], \text{E} \vdash_0^{\Omega \cdot 2 + \omega} A_j$. Hence k -fold application of (Cut) yields $f[c], \text{E} \vdash_{\Omega + d + 1}^{\Omega \cdot 3} A$, where $c := \max(\{k\} \cup \{c_j \mid j \leq k\} \cup \{\text{lh}(A_j) \mid 1 \leq j \leq k\})$ and $d := \max(\{\Omega, \text{rk}(A_1), \dots, \text{rk}(A_k)\})$.

For each $n \in \mathbb{N}$ and $\alpha \in \mathcal{OT}(\mathcal{F})$ let us define ordinal $\Omega_n(\alpha)$ and γ_n by

$$\begin{aligned}\Omega_0(\alpha) &= \alpha, & \gamma_0 &= \Omega \cdot 3, \\ \Omega_{n+1}(\alpha) &= \Omega^{\Omega_n(\alpha)}, & \gamma_{n+1} &= E^{\gamma_n}(0) + 1.\end{aligned}$$

Then d -fold iteration of Cut-reduction lemma (Lemma 25) yields the sequent $f[c]^{\gamma_d}, E \vdash_{\Omega+1}^{\Omega_d(\Omega \cdot 3)} A$. Hence Impredicative cut-elimination lemma (Lemma 30) yields

$$(f[c]^{\gamma_d})^{E^{\Omega_d(\Omega \cdot 3)}(0)}, E^{\Omega_d(\Omega \cdot 3)+1} \vdash_{\cdot}^{E^{\Omega_d(\Omega \cdot 3)}(0)} A.$$

Let $F := E^{\Omega_d(\Omega \cdot 3)+1}$ and $\beta := E^{\Omega_d(\Omega \cdot 3)}(0)$. Then $(f[c]^{\gamma_d})^\beta, F \vdash_{\omega^\beta}^\beta A$ holds. It is not difficult to check that $\beta < \Omega$, $N(\beta) \leq (f[c]^{\gamma_d})^\beta$ and $K_\Omega \beta < F(0)$. Hence Predicative cut-elimination lemma (Lemma 27) yields the sequent

$$(f[c]^{\gamma_d})^{F^{\Omega \cdot \beta + \beta \cdot 2}(0)+1} F \vdash_0^{\varphi \beta \beta} A.$$

Now let f denote s^ω . By Example 18.4 one can check that the conditions $(s^\omega.1)$ and $(s^\omega.2)$ hold. From Example 18 one will also see that $s^\omega[c](m) \leq s^\omega(s^c(m)) \leq s^{\omega+c+1}(m)$ for all m . By these we have the inequality

$$(s[c]^{\gamma_d})^{F^{\Omega \cdot \beta + \beta \cdot 2}(0)+1}(0) \leq ((s^{\omega+c+1})^{\gamma_d})^{F^{\Omega \cdot \beta + \beta \cdot 2}(0)+1}(0).$$

Thanks to Lemma 20 we can find an ordinal $\alpha \in \mathcal{OT}(\mathcal{F}) \upharpoonright \Omega$ built up without the Veblen function symbol φ such that

$$((s^{\omega+c+1})^{\gamma_d})^{F^{\Omega \cdot \beta + \beta \cdot 2}(0)+1}(0) \leq s^\alpha(0).$$

This together with (l -fold application of) Inversion lemma (Lemma 24) yields the sequent

$$s^\alpha[m_0] \cdots [m_{l-1}], F \vdash_0^{\varphi \beta \beta} \exists y B(\underline{\mathbf{m}}, y),$$

where $\mathbf{m} = m_0, \dots, m_{l-1}$. By Witnessing lemma (Lemma 31) we can find a natural $n \leq s^\alpha[m_0] \cdots [m_{l-1}](0) = s^\alpha(m_0 + \dots + m_{l-1})$ such that $B(\mathbf{m}, n)$ is true in the standard model \mathbb{N} of PA. \square

We say a function f is *elementary* (in another function g) if f is definable explicitly from the successor s , projection, zero 0 , addition $+$, multiplication \cdot , cut-off subtraction $\dot{-}$ (and g), using composition, bounded sums and bounded products, c.f. Rose [15, page 3].

Corollary 38. *Every function provably computable in \mathbf{ID}_1 is elementary in $\{s^\alpha \mid \alpha \in \mathcal{OT}(\mathcal{F}) \upharpoonright \Omega\}$.*

6 A recursive ordinal notation system $\mathcal{O}(\Omega)$

In order to obtain a precise characterisation of the provably computable functions of \mathbf{ID}_1 , we introduce a *recursive* ordinal notation system $\langle \mathcal{O}(\Omega), < \rangle$. Essentially $\mathcal{O}(\Omega)$ is a subsystem of $\mathcal{OT}(\mathcal{F})$.

Definition 39. We define three sets $\mathbf{SC} \subseteq \mathbb{H} \subseteq \mathcal{O}(\Omega)$ of ordinal terms simultaneously. Let 0 , Ω , \mathbf{S} , and $+$ be distinct symbols.

1. $0 \in \mathcal{O}(\Omega)$ and $\Omega \in \mathbf{SC}$.
2. If $\alpha \in \mathcal{OT}(\mathcal{F}) \upharpoonright \Omega$, then $\mathbf{S}(\alpha) \in \mathcal{O}(\Omega)$.
3. If $\{\alpha_1, \dots, \alpha_l\} \subseteq \mathbb{H}$ and $\alpha_1 \geq \dots \geq \alpha_l$, then $\alpha_1 + \dots + \alpha_l \in \mathcal{O}(\Omega)$.
4. If $\alpha \in \mathcal{O}(\Omega)$, then $\omega^\alpha \in \mathbb{H}$.
5. If $\alpha \in \mathcal{O}(\Omega)$ and $\xi \in \mathcal{O}(\Omega) \upharpoonright \Omega$, then $\Omega^\alpha \cdot \xi \in \mathbb{H}$.
6. If $\alpha \in \mathcal{O}(\Omega)$ and $\xi \in \mathcal{O}(\Omega) \upharpoonright \Omega$, then $\mathbf{S}^\alpha(\xi) \in \mathbf{SC}$.

The relation $<$ on $\mathcal{O}(\Omega)$ is defined in the obvious way. One will see that $\mathcal{O}(\Omega)$ is indeed a recursive ordinal notation system. Let us define the norm $N(\omega^\alpha)$ of ω^α in the most natural way, i.e., $N(\omega^\alpha) = N(\alpha) + 1$.

Lemma 40. Let α denote an ordinal term built up in $\mathcal{OT}(\mathcal{F})$ without the Veblen function symbol φ . Then there exists an ordinal term $\alpha' \in \mathcal{O}(\Omega)$ such that $\alpha \leq \alpha'$ and $N(\alpha) \leq N(\alpha')$.

Proof. By induction over the term construction of $\alpha \in \mathcal{OT}(\mathcal{F})$. In the base case let us observe that $\mathbf{E}(\alpha) \leq \mathbf{S}^1(\alpha)$ for all $\alpha < \Omega$ and that $N(\mathbf{E}(\alpha)) = N(\alpha) + 1 < N(\mathbf{S}(\alpha)) + 1 = N(\mathbf{S}^1(\alpha))$. In the induction case we employ Lemma 11. \square

Lemma 41. For any ordinal term $\alpha \in \mathcal{OT}(\mathcal{F})$ built up without the Veblen function symbol φ there exists an ordinal term $\alpha' \in \mathcal{O}(\Omega)$ such that $\mathbf{s}^\alpha(m) \leq \mathbf{s}^{\alpha'}(m)$ for all m .

Corollary 42. A function is provably computable in \mathbf{ID}_1 if and only if it is elementary in $\{\mathbf{s}^\alpha \mid \alpha \in \mathcal{O}(\Omega) \upharpoonright \Omega\}$.

The “only if” direction follows from Corollary 38 and Lemma 41. The “if” direction can be seen as follows. One can show that for each $\alpha \in \mathcal{O}(\Omega) \upharpoonright \Omega$ the system \mathbf{ID}_1 proves that the initial segment $\langle \mathcal{O}(\Omega) \upharpoonright \alpha, < \rangle$ of $\langle \mathcal{O}(\Omega), < \rangle$ is a well-ordering. For the full proof, we kindly refer the readers to, e.g., Pohlers [12, §29]. From this one can show that for each $\alpha \in \mathcal{O}(\Omega) \upharpoonright \Omega$ the function \mathbf{s}^α is provably computable in \mathbf{ID}_1 , and hence the assertion.

7 Conclusion

In this technical report we introduce a new approach to provably computable functions, providing a simplified characterisation of those of the system \mathbf{ID}_1 of non-iterated inductive definitions. The simplification is made possible due to the method of operator-controlled derivations that was originally introduced by Wilfried Buchholz [6]. An new idea in this report is to combine the ordinal operators from [6] with the number-theoretic operators from [19], c.f. Definition 23. Ordinal operators contain information much enough to analyse Π_1^1 -consequences of the controlled derivations. In contrast, number-theoretic operators contain information much enough to analyse those Π_2^0 -consequences. It is not difficult to

generalise this approach to the system \mathbf{ID}_n of n -fold iterated inductive definitions. Then it is natural to ask whether this approach can be extended to stronger systems like fragments of Kripke-Platek set theories. Extension to strong fragments, e.g., the fragment KPM for recursively Mahlo universes or the fragment \mathbf{KPII}_3 for Π_3 -reflecting universes, is still a challenge.

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